On Establishing Classic Performance Measures for Reset Control Systems*

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Abstract. Reset controllers are linear controllers that reset some of their states to zero when their inputs reach a threshold. We are interested in their feedback connection with linear plants, and in this context, the objective of this paper is twofold. First, to motivate the use of reset control through theory, simulations and experiments, and secondly, to summarize some of our recent results which establish classic performance properties ranging from quadratic and BIBO stability to steady-state and transient performance.

1 Introduction

It is well-appreciated that Bode's gain-phase relationship [1] places a hard limitation on performance tradeoffs in linear, time-invariant (LTI) feedback control systems. Specifically, the need to minimize the open-loop high-frequency gain often competes with required high levels of low-frequency loop gains and phase margin bounds. Our focus on reset control systems is motivated by its potential to improve this situation as demonstrated theoretically in [2] and by simulations and experiments [3]-[5].

The basic concept in reset control is to reset the state of a linear controller to zero whenever its input meets a threshold. Typical reset controllers include the so-called Clegg integrator [6] and first-order reset element (FORE) [3]. The former is a linear integrator whose output resets to zero when its input crosses zero. The latter generalizes the Clegg concept to a first-order lag filter. In [6], the Clegg integrator was shown to have a describing function similar to the frequency response of a linear integrator but with only $38.1°$ phase lag.

Reset control action resembles a number of popular nonlinear control strategies including relay control [7], sliding mode control [8] and switching control [9]. A common feature to these is the use of a switching surface to trigger change in control signal. Distinctively, reset control employs the same (linear) control law on both sides of the switching surface. Resetting occurs when the system trajectory impacts this surface. This reset action can be alternatively viewed as the injection of judiciously-timed, state-dependent impulses into an otherwise LTI feedback system. This analogy is evident in

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the paper where we use impulsive differential equations; e.g., see [10] and [11], to model dynamics. Despite this relationship, we found existing theory on impulse differential equations to be either too general or broad to be of immediate and direct use. This connection to impulsive control helps to draw comparison to a body of control work [12] where impulses were introduced in an open-loop fashion to quash oscillations in vibratory systems. Finally, we would like to point other recent research and applications of reset control found in [13]-[15].

The objective of this paper is twofold. First, to motivate reset control through theory, simulations and experiments, and secondly, to summarize some of our recent results ([2], [16]-[20]) which establish properties ranging from quadratic and BIBO stability to steady-state and transient performance. The paper is organized as follows. The next section provides three examples to demonstrate the advantage in using reset control. After that, Section 3 writes out the dynamical equations of our reset control systems and in Section 4 we present one of our main results giving a necessary and sufficient conditions for quadratic stability. In Section 5 we give internal model and superposition principles. Specializing to first-order reset elements, we then go on in Section 6 to establish results concerning BIBO stability. We then restrict attention to a class of reset control systems whose linear dynamics are second-order dominant. For this classic situation, we will show that the associated reset control system is always stable, enjoys steady-state performance akin to its linear counterpart and can be designed for improved overshoot in its step response.

2 Motivation

In this section we give three examples comparing reset to linear feedback control. The first gives an example of control specifications not achievable by any linear feedback control, but achievable using reset. The second example shows how the simple introduction of reset in a control loop reduces step-response overshoot without sacrificing rise-time. Lastly, we describe an experimental setup of reset where we again demonstrate reset-control's potential.

2.1 Overcoming limitations of linear control

Consider the standard linear feedback control system in Figure 1 where the plant $P(s)$ contains an integrator. Assume that $C(s)$ stabilizes. In [21] it was shown that the tracking error e due to a unit-step input satisfies

$$\int_0^\infty e(t)dt = \frac{1}{K_v}$$

where the velocity constant K_v is defined by $K_v \triangleq lim_{s \to 0} sP(s)C(s)$. Alone, this constraint does not imply overshoot in the step response y; i.e., $y(t) \geq 1$

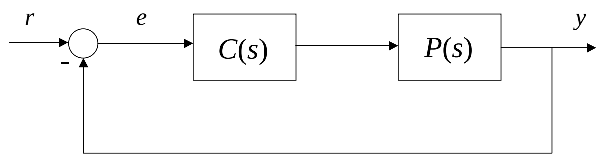

Fig. 1. Linear feedback control system.

for some $t > 0$. However, introduction of an additional, sufficiently stringent time-domain bandwidth constraint will. To see this, consider the notion of rise time t_r introduced in [21]:

$$t_r = \sup_T \left\{ T : y(t) \leq \frac{t}{T},\ t \in [0, T] \right\}.$$

The following result (see [2]) is quite immediate.

Fact: *If $t_r > \frac{2}{K_v}$; i.e., the rise time is sufficiently slow, then the unit-step response $y(t)$ overshoots.*

To illustrate this result consider the plant $P(s)$ in Figure 1 as a simple integrator. In addition to closed-loop stability suppose the design objectives are the following:

- Steady-state error no greater than 1 when tracking a unit-ramp input.
- Rise time greater than 2 seconds when tracking a unit-step.
- No overshoot in the step response.

To meet the error specification on the ramp response, this linear feedback system must have velocity error constant $K_v \geq 1$. Since $t_r > 2 \geq \frac{2}{K_v}$, the Fact indicates that no stabilizing $C(s)$ exists to meet all the above objectives. However, these specifications can be met using reset control with a first-order reset element (FORE) described by

$$\begin{aligned} \dot{u}_r(t) &= -bu_r(t) + e(t); \quad && e(t) \neq 0 \\ u_r(t^+) &= 0; && e(t) = 0 \end{aligned}$$

where b, the FORE's pole, is chosen as $b = 1$. Indeed, in Section 6.2 of this paper we will show that this reset system is asymptotically stable, has

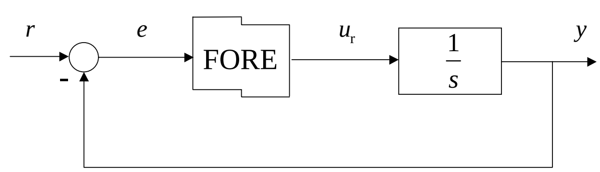

Fig. 2. Reset control of an integrator using a first-order reset element.

bounded response y to bounded input r and zero steady-state tracking error e to constant r. This reset control system is given in Figure 2. Figure 3 shows a simulation of this control system's tracking error e to a unit-ramp input. The steady-state error is one. In Figure 4 we show its response y to a unit-step input and see that its rise time t_r is greater than 2 seconds and has no overshoot[1]. Thus, this reset control system meets the previously stated design objectives that were not attainable using linear feedback control.

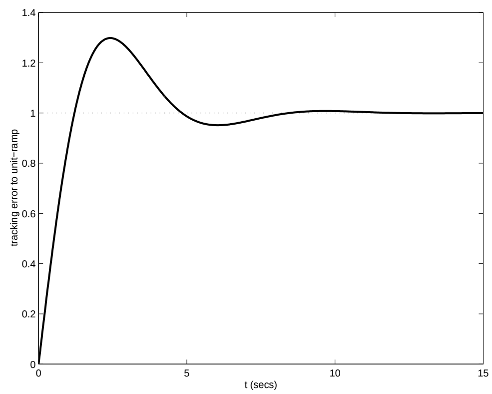

Fig. 3. Tracking-error e to a unit-ramp input for the reset control system.

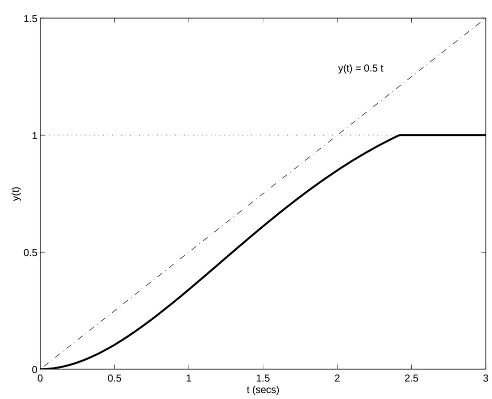

Fig. 4. Output response y to a unit-step input for the reset control system.

[1] The step response in Figure 4 is deadbeat. This occurs since $(u, y) = (0, r)$ is an equilibrium point.

2.2 Reducing overshoot

Another motivation to use reset control is that it provides a simple means to reduce overshoot in a step response. For example, consider the feedback system in Figure 5 where the loop transfer function is:

$$L(s) = \frac{1}{s(s+0.2)}$$

and where the FORE's pole is set to $b = 1$. Without reset, the linear closed-loop system has standard second-order transfer function

$$\frac{Y(s)}{R(s)} = \frac{1}{s^2 + 2(0.1)s + 1}.$$

The damping ratio is $\zeta = 0.1$ and the step response exhibits the expected 70%

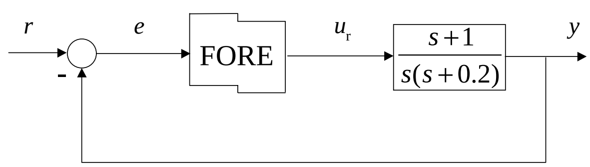

Fig. 5. Resetting can reduce overshoot in response to step reference inputs.

overshoot as shown in Figure 6. The step response of the reset control system is also shown in this figure and it has only 40% overshoot while retaining the rise time of the linear design. Moreover, as in the previous example, this reset control system can be shown to be asymptotically and BIBO stable, and to asymptotically track step inputs r; see Section 6.2. Also, the level of overshoot can be quantitatively linked to the FORE's pole b. This will also be discussed in Section 6.2. Thus, the performance of a classical second-order dominant feedback control system can be significantly improved through the *simple* introduction of reset control.

2.3 Demonstrating performance in the lab

The benefits of reset control have also been realized in experimental settings. Here we describe a laboratory setup in which we applied both linear and reset control to the speed control of the rotational flexible mechanical system shown in Figure 7. This system consists of three inertias connected via flexible shafts. A servo motor drives inertia J_3 and the speed of inertia J_1 is measured via a tachometer. The controller was implemented using dSPACE tools [22]. A more complete description of this experiment can be found in [19].

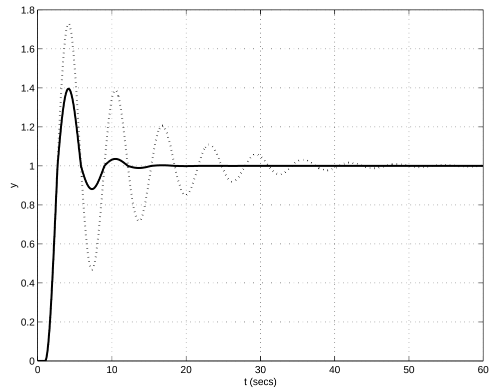

Fig. 6. Comparison of step responses between reset (solid) and the linear control system (dotted).

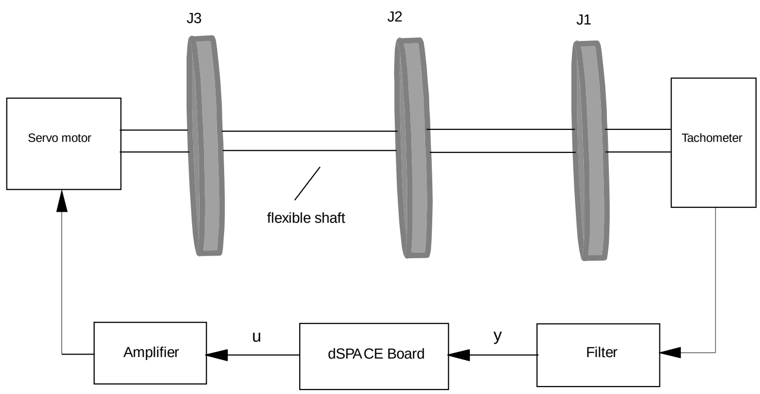

Fig. 7. Schematic of the rotational flexible mechanical system.

Tradeoffs in linear feedback control A block diagram of a linear feedback control system is shown in Figure 8 where the plant $P(s)$ was identified from frequency-response data of the flexible mechanical system as:

$$P(s) = \frac{46083950}{(s + 1.524)(s^2 + 3.1s + 2820)(s^2 + 3.62s + 9846)}.$$

We posed the following specifications to illustrate the limitations and tradeoffs in LTI design and their subsequent relief using reset control:

1. *Bandwidth constraint*: The unity-gain cross-over frequency ω_c, defined by $|PC(j\omega_c)| = 1$, must satisfy $\omega_c > 3\pi$.
2. *Disturbance rejection*: Low-frequency disturbances are to be rejected; specifically,

$$\left|\frac{y(j\omega)}{d(j\omega)}\right| \leq 0.2, \quad \text{when} \quad \omega \leq \pi;$$

3. *Sensor-noise suppression*: High-frequency sensor noise is to be suppressed; i.e.,

$$\left|\frac{y(j\omega)}{n(j\omega)}\right| \leq 0.3, \quad \text{when} \quad \omega \geq 10\pi;$$

4. *Asymptotic performance*: Zero steady-state tracking error to constant reference r and disturbance d signals.
5. *Overshoot*: Overshoot in output y to a constant reference r should be less than 20%.

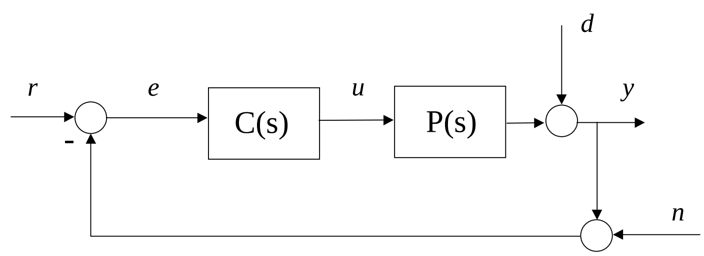

Fig. 8. The linear feedback control system.

In terms of Bode specifications, the first two constraints translate into minimum-gain requirements on the open-loop gain $|PC(j\omega)|$ at low frequencies while the third specification places an upper bound on this gain at high frequencies. The fourth specification requires $C(s)$ to contain an integrator and the fifth specification requires a phase margin of approximately 45°; assuming second-order dominance.

Using classical loop-shaping techniques we were unable to meet all of the above specifications. To illustrate the tradeoffs, consider two candidate, stabilizing LTI controllers:

$$C_1(s) = \frac{1281489(s + 4.483)(s^2 + 3.735s + 2851)(s^2 + 5.158s + 10060)}{s(s^2 + 295.1s + 22330)(s^2 + 126.2s + 8889)(s^2 + 239s + 27560)}$$

and

$$C_2(s) = \frac{1075460(s + 7)(s^2 + 3.662s + 2798)(s^2 + 5.419s + 9876)}{s(s + 209.6)(s + 35.8)(s^2 + 132.8s + 12050)(s^2 + 375.9s + 66930)}.$$

Figure 9 compares the Bode plots of the corresponding loops $L_1(j\omega) = P(j\omega)C_1(j\omega)$ and $L_2(j\omega) = P(j\omega)C_2(j\omega)$. Loop L_1 fails to satisfy the sensor-noise suppression specification at $\omega = 10\pi$. This specification can be met by reducing the gain of $L_1(j\omega)$ as done with $L_2(j\omega)$. This is verified by the time response y to 5 Hz sinusoidal noise n in Figure 10. Since both designs stabilize and since both low-frequency gains are constrained by the first two

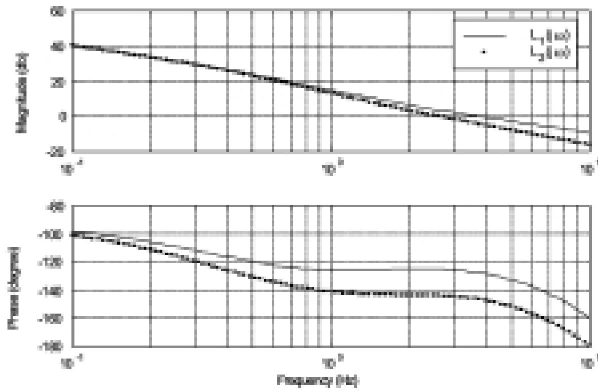

Fig. 9. Bode plots of L_1 and L_2.

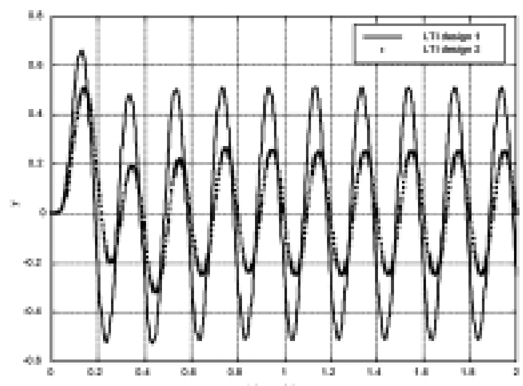

Fig. 10. Comparison between LTI designs L_1 and L_2 of output y to $n(t) = \sin(10\pi t)$.

specifications, Bode's gain-phase relationship [1] dictates that $L_2(j\omega)$ must have correspondingly larger phase lag as verified in the phase plot of Figure 9. The reduced gain in $L_2(j\omega)$ comes at the expense of a smaller phase margin and hence larger overshoot as shown in the step responses in Figure 11. Extensive tuning of these controllers failed to yield a design meeting all specifications.

Reset control design Now we turn to reset control design where we exploit its potential to satisfy the above specifications. The design procedure consists of two steps as developed in [3]-[5]. First, we design a linear controller to meet all the specifications - except for the overshoot constraint; $C_2(s)$ is a suitable choice. The second step is to select the FORE's pole b to meet the overshoot specification. In this respect, [Figure 5, 3] provides a guideline for this choice. Using this tool, we select $b = 14$. The resulting reset control system is shown

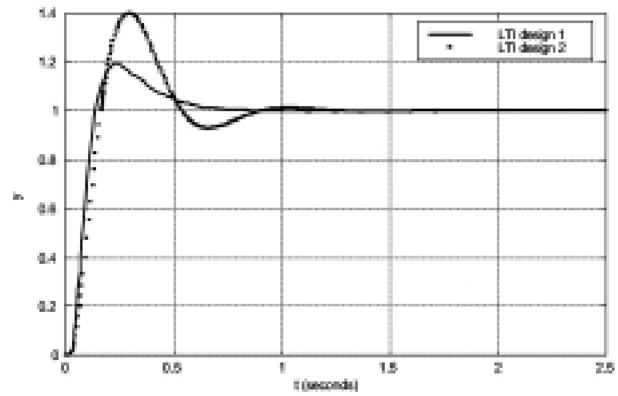

Fig. 11. Comparison between LTI designs L_1 and L_2 of output y to $r(t) \equiv 1$.

in Figure 12. Later in this paper we show that this reset control system is quadratically and BIBO stable and asymptotically tracks constant reference inputs r.

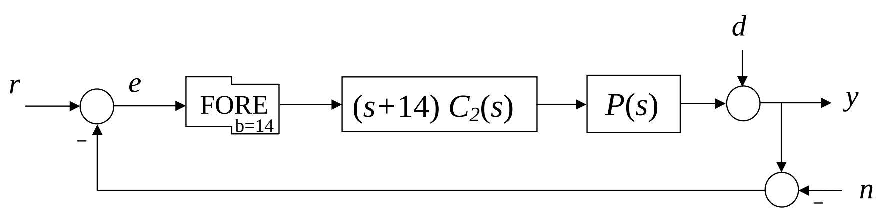

Fig. 12. Reset control system for the flexible mechanism.

Finally, we compare the performance of the LTI (using L_1) and reset control systems. Figures 13 and 14 show that the reset control system has better sensor-noise suppression to a 5 Hz sinusoid and to white-noise.

However, unlike the LTI tradeoff experienced by controller $C_2(s)$, the reset control system has comparable transient response as shown in Figure 15[2].

[2] The steady-state noise in Figure 15 is due to ripple in the the tach-generator.

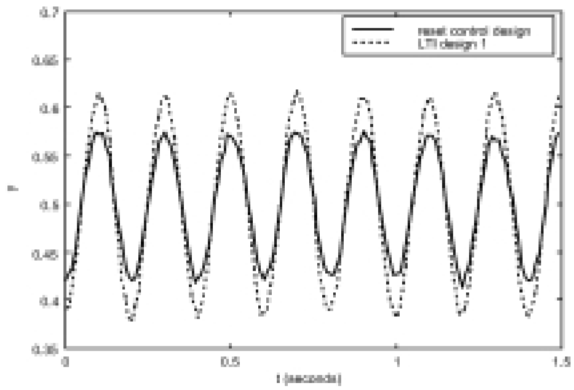

Fig. 13. Comparison of steady-state response y to $r(t) \equiv 1$ and $n(t) = \sin(10\pi t)$.

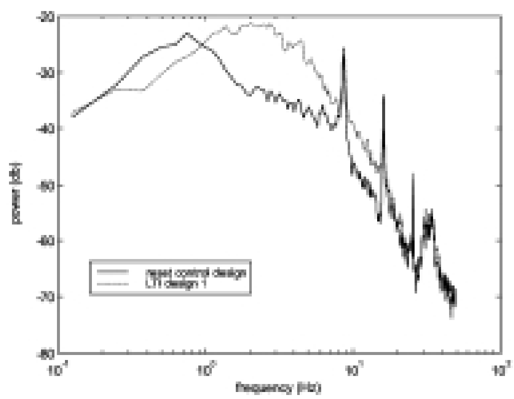

Fig. 14. Comparison of output y power spectra when n is white sensor noise.

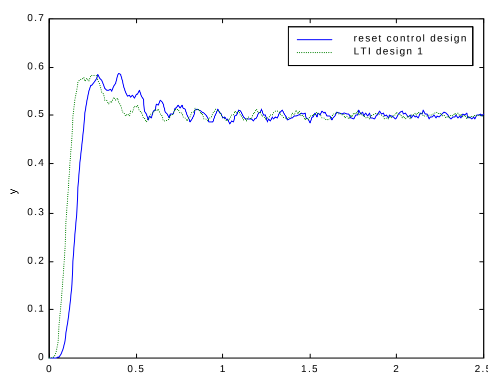

Fig. 15. Comparison between reset and LTI control (using L_1) of output y to $r(t) \equiv 1$.

3 The Dynamics of Reset Control Systems

The reset control system considered in this paper is shown in Figure 16 where the reset controller R is described by the impulsive differential equation (IDE) (see [10])

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r e(t); && e(t) \neq 0 \\ x_r(t^+) &= A_{Rr} x_r(t); && e(t) = 0 \\ u_r(t) &= C_r x_r(t) \end{aligned} \tag{1}$$

where $x_r(t) \in \mathbb{R}^{n_r}$ is the reset controller state and $u_r(t) \in \mathbb{R}$ is its output. The matrix $A_{Rr} \in \mathbb{R}^{n_r \times n_r}$ identifies that subset of states x_r that are reset. For example, in this paper we will assume that the last n_{rr} states x_{rr} are reset and use the structure $A_{Rr} = \begin{bmatrix} I_{n_r - n_{rr}} & 0 \\ 0 & 0 \end{bmatrix}$. Illustrations of (1) include the Clegg integrator described by

$$A_r = 0; \quad B_r = 1; \quad C_r = 1; \quad A_{Rr} = 0$$

and the FORE having

$$A_r = b; \quad B_r = 1; \quad C_r = 1; \quad A_{Rr} = 0. \tag{2}$$

The linear controller $C(s)$ and plant $P(s)$ have, respectively, state-space realizations:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c u_r(t) \\ u_c(t) &= C_c x_c(t) \end{aligned}$$

and

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u_c(t) \\ y(t) &= C_p x_p(t) \end{aligned}$$

where $x_c(t) \in \mathbb{R}^{n_c}$, $x_p(t) \in \mathbb{R}^{n_p}$ and $y(t) \in \mathbb{R}$. The closed-loop system can then be described by the IDE

$$\begin{aligned} \dot{x}(t) &= A_{c\ell} x(t); \quad x(t) \notin \mathcal{M}; \quad x(0) = x_0 \\ x(t^+) &= A_R x(t); \quad x(t) \in \mathcal{M} \\ y(t) &= C_{c\ell} x(t) \end{aligned} \tag{3}$$

where

$$x \triangleq \begin{bmatrix} x_p \\ x_c \\ x_r \end{bmatrix}; \quad A_{c\ell} \triangleq \begin{bmatrix} A_p & B_p C_c & 0 \\ 0 & A_c & B_c C_r \\ -B_r C_p & 0 & A_r \end{bmatrix};$$

$$A_R \triangleq \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & A_{Rr} \end{bmatrix}; \quad C_{c\ell} \triangleq \begin{bmatrix} C_p & 0 & 0 \end{bmatrix}$$

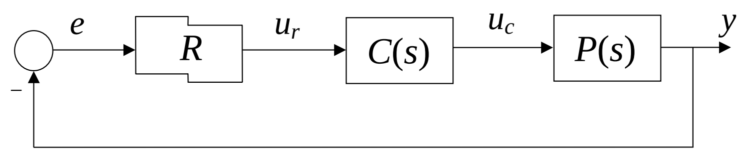

Fig. 16. Block diagram of a reset control system.

and where the *reset surface* $\mathcal{M}$ is the set of states for which $e = 0$. More precisely,

$$\mathcal{M} \triangleq \{\xi : C_{c\ell}\xi = 0;\ \ (I - A_R)\,\xi \neq 0\}.$$

As a consequence of this definition,

$$x(t) \in \mathcal{M} \quad \Rightarrow \quad x(t^+) \notin \mathcal{M}.$$

The times $t = t_i$ at which the system trajectory x intersects the reset surface $\mathcal{M}$ are referred to as *reset times*. These instants depend on initial-conditions and are collected in the ordered set:

$$\mathcal{T}(x_o) \triangleq \{t_i : t_i < t_{i+1}; x(t_i) \in \mathcal{M}, i = 1, 2, \ldots, \infty\}.$$

The solution to (3) is piecewise left-continuous on the intervals $(t_i, t_{i+1}]$. We define the *reset intervals* τ_i by

$$\tau_1 \triangleq t_1;$$
$$\tau_{i+1} \triangleq t_{i+1} - t_i, \quad i \in \mathbb{N}.$$

We make the following assumption on the set of reset times:

Resetting Assumption: *Given initial condition $x_0 \in \mathbb{R}^n$, the set of reset times $\mathcal{T}(x_o)$ is an unbounded, discrete subset of $\mathbb{R}_+$.*

Unboundedness of the set of reset times implies continual resetting. If this condition is not satisfied, then, after the last reset instance, the reset control system behaves as its base-linear system. We avoid these trivial cases. Discreteness of $\mathcal{T}(x_0)$, together with this unboundedness, guarantees the existence and continuation of solutions to (3). Finally, in absence of resetting; i.e., when $A_R = I$, the resulting linear system is called the *base-linear* system. We denote the loop, sensitivity and complementary sensitivity transfer functions of the base-linear system by:

$$L(s) = P(s)C(s)R(s), \quad S(s) = \frac{1}{1 + L(s)}, \quad T(s) = \frac{L(s)}{1 + L(s)}$$

where $R(s)$ is the transfer of (1) when $A_R = I$.

4 Quadratic Stability

In this section we give a necessary and sufficient condition for (3) to possess a quadratic Lyapunov function. First, we state some general Lyapunov-like stability conditions for our reset control systems which are similar to the analysis in [10] and [23]. Their proofs are relegated to the Appendix. As usual, $\dot{V}$ is the time-derivative of a Lyapunov candidate $V(x)$ along solutions, while $\Delta V \overset{\triangle}{=} V(x) - V(A_R x)$, is the jump in $V(x)$ when the trajectory strikes $\mathcal{M}$.

Proposition 1: (Local Stability) *Let Ω be an open neighborhood of the equilibrium point $x = 0$ of* (3) *and let $V(x) : \Omega \to \mathbb{R}$ be a continuously-differentiable, positive-definite function such that*

$$\dot{V}(x) \leq 0; \quad x \in \Omega / \mathcal{M} \tag{4}$$

$$\Delta V(x) \leq 0; \quad x \in \Omega \cap \mathcal{M}. \tag{5}$$

Then, under the Resetting Assumption, $x = 0$ is locally stable. Moreover, if either

$$\dot{V}(x) < 0; \quad x \in \Omega / \{0\} \tag{6}$$

or

$$\Delta V(x) < 0; \quad x \in \Omega \cap \mathcal{M}, \tag{7}$$

then $x = 0$ is asymptotically stable.

Proposition 2: (Global Stability) *Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a continuously-differentiable, positive-definite, radially-unbounded function such that*

$$\dot{V}(x) \leq 0; \quad x \notin \mathcal{M}$$

$$\Delta V(x) \leq 0; \quad x \in \mathcal{M}.$$

Then, under the Resetting Assumption, $x = 0$ is stable. Moreover, if either

$$\dot{V}(x) < 0; \quad x \in \mathbb{R}^n / \{0\},$$

or

$$\Delta V(x) < 0; \quad x \in \mathcal{M},$$

then $x = 0$ is globally asymptotically stable.

We now specialize to quadratic Lyapunov functions.

Definition: The reset control system (3) is said to be *quadratically stable* if there exists a positive-definite symmetric matrix P such that $V(x) = x'Px$ satisfies the asymptotic stability conditions of Proposition 2.

Theorem 1: *Under the Resetting Assumption, the reset control system described in* (3) *is quadratically stable if and only if there exists a* $\beta \in \mathbb{R}^{n_{rr}}$ *such that*

$$H_\beta(s) \triangleq [\beta C_p \;\; 0 \;\; I_{n_{rr}}]\,(sI - A_{c\ell})^{-1} \begin{bmatrix} 0 \\ I_{n_{rr}} \end{bmatrix} \tag{8}$$

is strictly positive real (*SPR*)[3].

Proof: (Sufficiency) We first define $V(x) = x'Px$. By Proposition 2 the reset control system described in (3) is quadratically stable if there exists a positive-definite symmetric matrix P such that

$$x'\,(A'_{c\ell}P + PA_{c\ell})\,x < 0; \quad x \in \mathbb{R}^n / \{0\} \tag{9}$$

and

$$x'\,(A'_R P A_R - P)\,x \le 0; \quad x \in \mathcal{M}. \tag{10}$$

Let Θ be a matrix whose such that its columns span the nullspace of $C_{c\ell}$. Using this, we can express

$$\mathcal{M} = \{\Theta\xi : (I - A_R)\,\Theta\xi \ne 0; \;\; \xi \ne 0\}$$

and define $\bar{\mathcal{M}}$ as

$$\bar{\mathcal{M}} \triangleq \left\{\Theta\xi : \xi \in \mathbb{R}^{n-n_{rr}}\right\} \supset \mathcal{M}.$$

Therefore,

$$\Theta'\,(A'_R P A_R - P)\,\Theta \le 0 \tag{11}$$

implies that (10) holds. (11) holds for some positive-definite P if there exists a $\beta \in \mathbb{R}^{n_{rr}}$ such that

$$[0 \;\; I_{n_{rr}}]P = [\beta \;\; C_p \;\; 0 \;\; I_{n_{rr}}]. \tag{12}$$

Thus, the proof reduces to finding a positive-definite symmetric matrix P such that (9) and (12) hold. From the Kalman-Yakubovich-Meyer (KYM) lemma; e.g., see [24], such P exists if there exists a $\beta \in \mathbb{R}^{n_{rr}}$ such that $H_\beta(s)$ in (8) is SPR.

(Necessity) Suppose (3) is quadratically stable. Then, from Proposition 2 there exists a positive-definite symmetric matrix P such that

$$x'(A'_{c\ell}P + PA_{c\ell})x < 0; \qquad x \ne 0 \tag{13}$$

$$x'(A'_R P A_R - P)x \le 0; \qquad x \in \mathcal{M}. \tag{14}$$

[3] A transfer function $X(s)$ is said to be *strictly positive real* if $X(s)$ is asymptotically stable, and $\mathrm{Re}[X(j\omega)] > 0, \;\; \forall\omega \ge 0$.

The continuity of ΔV, together with (14), implies that

$$x'(A_R'PA_R - P)x \leq 0; \qquad x \in \bar{\mathcal{M}}$$

which in turn implies that (12) holds for some $\beta \in \mathbb{R}^{n_{rr}}$. The strict positive realness of H_β in (8) then follows from (12), (13) and the KYM lemma. This concludes the proof.

Theorem 1 gives an easily-testable condition for the quadratic stability of the reset control systems described by (3). This condition is also key in showing that reset control systems enjoy other properties. Before we present these results, we formally introduce first-order reset elements.

5 Steady-state performance

In this section we study the steady-state performance of the reset control system in (3) and show that it enjoys an internal model principle and steady-state superposition property.

5.1 An internal model principle

We introduce an *internal model principle* for the reset control systems by considering a model of the reference signal r inside the loop as part of $P(s)C(s)$. We can state the following theorem.

Theorem 2: *Under the Resetting Assumption, if $P(s)C(s)$ contains an internal model of r, and if there exists a $\beta \in \mathbb{R}^{n_{rr}}$ such that $H_\beta(s)$ in* (8) *is SPR, then the reset control system described in* (3) *achieves asymptotic tracking of the reference input r.*

Proof: We first adopt the realization $\{A_{pc}, B_{pc}, C_{pc}\}$ for $P(s)C(s)$, which contains an internal model of the reference input. Since r is realized within $P(s)C(s)$, there exists states $z(t)$ and $C_{zr} \in \mathbb{R}^{n_p+n_c}$, such that

$$\begin{aligned} \dot{z}(t) &= A_{pc}z(t); \quad z(0) = r(0), \\ r(t) &= C_{rz}z(t). \end{aligned}$$

Then using the state transformation $\tilde{x}_{pc}(t) \stackrel{\triangle}{=} x_{pc}(t) - z(t)$ we obtain the following:

$$\begin{aligned} \dot{\tilde{x}}(t) &= A_{c\ell}\tilde{x}(t); \quad \tilde{x}(t) \notin \mathcal{M}; \quad \tilde{x}(0) = x_0 - r(0), \\ \tilde{x}(t^+) &= A_R\tilde{x}(t); \quad \tilde{x}(t) \in \mathcal{M}, \\ y(t) &= C_{c\ell}\tilde{x}(t) + r(t), \end{aligned}$$

where $\tilde{x} = \begin{bmatrix} \tilde{x}_{pc} \\ x_{rr} \end{bmatrix}$. Hence, the asymptotic tracking problem is now expressed as the asymptotic stability problem for the unforced system. By Theorem 1, asymptotic stability of the unforced system is guaranteed if there exists a $\beta \in \mathbb{R}^{n_{rr}}$ such that $H_\beta(s)$ is SPR. This concludes the proof.

5.2 A superposition principle

We now introduce an additional input to the control system as shown in the Figure 17. This system can be described by

$$\begin{aligned} \dot{x}(t) &= A_{c\ell}x(t) + B_{c\ell}r_1(t) + B_{c\ell}r_2(t); && x(t) \notin \mathcal{M}; \quad x(0) = x_0 \\ x(t^+) &= A_R x(t); && x(t) \in \mathcal{M}, \\ y(t) &= C_{c\ell}x(t) && \end{aligned} \tag{15}$$

where

$$\mathcal{M} \triangleq \{\xi : r_1(t) + r_2(t) - C_{c\ell}\xi = 0; \;\; (I - A_R)\,\xi \neq 0\}.$$

The next result gives a steady-state superposition result. If the loop contains an internal model of one of the inputs signals, say r_2, then this result claims that the steady-state response to $r_1 + r_2$ is simply the steady-state response to r_1.

Corollary 1: *Consider the reset control system with two inputs* r_1 *and* r_2 *described in* (15)*. Suppose the Resetting Assumption is in force,* $P(s)C(s)$ *contains an internal model of* r_2 *and there exists a* $\beta \in \mathbb{R}^{n_{rr}}$ *such that* $H_\beta(s)$ *is strictly positive real (SPR). Then, the steady-state error,* $\lim_{t\to\infty} e(t)$ *is independent of* r_2*.*

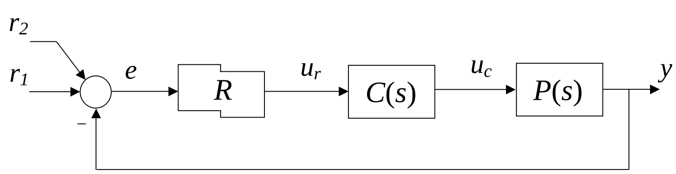

Fig. 17. Block diagram of a reset control system with two inputs.

6 Specialization to First-Order Reset Elements

As illustrated in Section 2.3 the design of reset control systems, as developed in [3] and [5], involves the synthesis of both linear compensator $C(s)$ and reset controller R in Figure 16. Typically, $C(s)$ is designed to stabilize the base-linear system and shape the loop $L(s) = P(s)C(s)R(s)$ to satisfy classical Bode specifications at high and low frequencies. The reset controller is then designed to meet overshoot specifications. In this subsection we focus on FORE described by (1) and (2) where $b > 0$ is the FORE's pole.

6.1 BIBO stability

Consider the reset control system with a reference input as shown in Figure 18 and described by the following IDE

$$\begin{aligned} \dot{x}(t) &= A_{c\ell}x(t) + B_{c\ell}r(t); \quad && x(t) \notin \mathcal{M}; \quad x(0) = 0, \\ x(t^+) &= A_R x(t); && x(t) \in \mathcal{M}, \\ y(t) &= C_{c\ell}x(t) \end{aligned} \tag{16}$$

where

$$B_{c\ell} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\mathcal{M} \stackrel{\triangle}{=} \{\xi : r(t) - C_{c\ell}\xi = 0; \;\; (I - A_R)\,\xi \neq 0\}.$$

In this section we analyze the BIBO stability of (16) which requires every

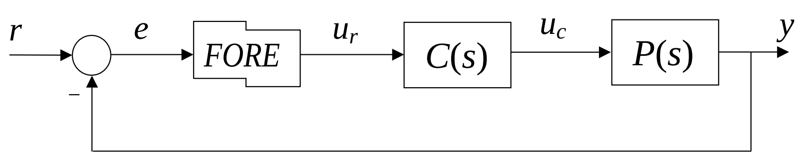

Fig. 18. Block diagram of a reset control system with reference input r.

bounded input[4] r to produce a bounded output y. To begin this analysis we let x_ℓ be the state of the base-linear system; that is:

$$\dot{x}_\ell(t) = A_{c\ell}x_\ell(t) + B_{c\ell}r(t); \quad x(0) = 0$$

and take $z \stackrel{\triangle}{=} x - x_\ell$. We partition

$$x = \begin{bmatrix} x_L \\ x_r \end{bmatrix}; \quad z = \begin{bmatrix} z_L \\ z_r \end{bmatrix}; \quad x_\ell = \begin{bmatrix} x_{\ell L} \\ x_{\ell r} \end{bmatrix}.$$

Applying the transformations:

$$\begin{aligned} z_L(t) &= x_L(t) - x_{\ell L}(t); \\ z_r(t) &= x_r(t) - x_{\ell r}(t) \end{aligned}$$

to (16) and expressing the reset rule with the set of reset times $\mathcal{T}(0)$ we obtain:

$$\begin{aligned} \dot{z}_L(t) &= Az_L(t) + Bz_r(t) \\ \dot{z}_r(t) &= -Cz_L(t) - bz_r(t); \qquad && t \notin \mathcal{T}(0) \\ z_r(t_i^+) &= -x_{\ell r}(t_i); && t \in \mathcal{T}(0). \end{aligned} \tag{17}$$

[4] A signal z is said to *bounded* if there exists a constant M such that $|z(t)| < M$ for all t.

As an intermediate step, we show that boundedness of z_L implies that y is bounded.

Lemma 1: *Assume $A_{c\ell}$ is asymptotically stable and r is bounded. If z_L is bounded, then output y is bounded.*

Proof: Suppose z_L is bounded. We have

$$\begin{aligned} |y(t)| &= |Cx_L(t)| \\ &\leq |Cz_L(t)| + |Cx_{\ell L}(t)| \,. \end{aligned}$$

Since $A_{c\ell}$ is stable and r is bounded, then $x_{\ell L}$ is bounded. Output y is thus bounded.

Before establishing BIBO stability, we need the following lemma.

Lemma 2: *If $A_{c\ell}$ is asymptotically stable and r is bounded, there exist constants M_1 and M_2 such that $|z_r(t_i^+)| < M_1$ and $|Cz_L(t_i)| < M_2$ for $i = 1, 2, \ldots, \infty$.*

Proof: Because $A_{c\ell}$ is asymptotically stable and r, then $x_{\ell r}$ and $x_{\ell L}$ are bounded. From (17), $z_r(t_i^+) = -x_{\ell r}(t_i)$. Therefore, there exists an M_1 such that $|z_r(t_i^+)| < M_1$ for $i = 1, 2, \ldots, \infty$. By definition, $Cz_L(t_i) = r(t_i) - Cx_{\ell L}(t_i)$. Since r and $x_{\ell L}$ are bounded, then there exists an M_2 such that $|Cz_L(t_i)| < M_2$ for $i = 1, 2, \ldots, \infty$.

We can now state our BIBO stability result.

Theorem 3: *Under the Resetting Assumption the reset control system* (16) *is BIBO stable if it is quadratically stable; i.e., there exists a $\beta \in \mathbb{R}$ such that $H_\beta(s)$ in* (8) *is SPR.*

Proof: Since $H_\beta(s)$ in (8) is strictly positive-real, then, from the KYM lemma, there exists a positive-definite matrix P, a vector q and a positive constant ε such that

$$\begin{aligned} &PA_{cl} + A_{cl}'P = -q'q - \varepsilon P; \\ &P[0 \ \cdots \ 0 \ 1]' = [\beta C \quad 1]'. \end{aligned} \tag{18}$$

Hence, P can be written as

$$P = \begin{bmatrix} P_1 & \beta C' \\ \beta C & 1 \end{bmatrix}$$

where $P_1 \in \mathbb{R}^{n \times n}$ is positive-definite. Along the piecewise left-continuous solutions of (17) we define

$$\begin{aligned} V(t) &= [z_L'(t), z_r(t)]P[z_L'(t), z_r(t)]' \\ &= z_L'(t)P_1 z_L(t) + 2\beta C z_L(t) z_r(t) + z_r^2(t) \end{aligned}$$

over $t \in (t_i, t_{i+1}]$. At the reset instants $t = t_i$ we then have

$$\begin{aligned} V(t_i^+) &= z_L'(t_i)P_1 z_L(t_i) + 2\beta C z_L(t_i) z_r(t_i^+) + z_r^2(t_i^+) \\ &= V(t_i) + 2\beta C z_L(t_i) z_r(t_i^+) + z_r^2(t_i^+) - 2\beta C z_L(t_i) z_r(t_i) - z_r^2(t_i). \end{aligned}$$

Since $-2\beta C z_L(t_i) z_r(t_i) - z_r^2(t_i) \le (\beta C z_L(t_i))^2$,

$$\begin{aligned} V(t_i^+) &\le V(t_i) + 2\beta C z_L(t_i) z_r(t_i^+) + z_r^2(t_i^+) + (\beta C z_L(t_i))^2 \\ &= V(t_i) + [z_r(t_i^+) + \beta C z_L(t_i)]^2. \end{aligned} \tag{19}$$

Because r is bounded, it follows from Lemma 2 that there exists a constant $M > 0$ such that $[z_r(t_i^+) + \beta C z_L(t_i)]^2 \le M$ for $i = 1, 2, \ldots, \infty$. Thus, from (19):

$$V(t_i^+) \le V(t_i) + M, \quad i = 1, 2, \ldots, \infty. \tag{20}$$

Differentiating $V(t)$ along solutions to (17), we use (18) to obtain

$$\begin{aligned} \dot{V}(t) &= [z_L'(t), z_r(t)](PA_{c\ell} + A_{c\ell}'P)[z_L'(t), z_r(t)]' \\ &= [z_L'(t), z_r(t)](-q'q - \varepsilon P)[z_L'(t), z_r(t)]' \\ &\le -\varepsilon[z_L'(t), z_r(t)]P[z_L'(t), z_r(t)]' \\ &= -\varepsilon V(t) \end{aligned}$$

for all $t \in (t_i, t_{i+1}]$. The non-negativity of $V(t)$ implies

$$V(t) \le e^{-\varepsilon(t - t_i)} V(t_i^+) \tag{21}$$

whenever $t \in (t_i, t_{i+1}]$. Since $t_{i+1} - t_i > \sigma$,

$$\begin{aligned} V(t_{i+1}) &\le e^{-\varepsilon(t_{i+1} - t_i)} V(t_i^+) \\ &\le e^{-\varepsilon\sigma} V(t_i^+) \\ &\le e^{-\varepsilon\sigma}[V(t_i) + M]. \end{aligned}$$

Combining (20) with (21) and repeatedly applying the above gives

$$V(t) \le e^{-\varepsilon(t - t_i)}[e^{-\varepsilon i\sigma} V(0) + M + e^{-\varepsilon\sigma} M + \ldots + e^{-\varepsilon i\sigma} M]$$

for all $t \in (t_i, t_{i+1}]$. Since $V(0) = 0$, $V(t) \le M/(1 - e^{-\varepsilon\sigma})$. Therefore, V is bounded. Because P is positive-definite, it follows that z_L is bounded. Finally, from Lemma 1, y is bounded. This completes the proof.

6.2 When the base-linear system has classical second-order form

In this section we focus on a class of first-order reset control systems shown in Figure 17 where

$$P(s)C(s) = \frac{(s + b)\omega_n^2}{s(s + 2\zeta\omega_n)}. \tag{22}$$

As a result, the associated base-linear system has classical second-order (complementary sensitivity) transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

This setup allows us to compare the reset control system's performance against a linear control system with dominant pole-pair. We will show that this class of reset control systems is always quadratically stable and, by virtue of Theorem 3, BIBO stable. We will characterize the step response of the reset control system in terms of standard measures such as rise-time, overshoot and settling time, thus allowing a direct comparison to its base-linear system. First, we establish that the key SPR condition in (8) is always satisfied.

***H_β* is always positive-real** We begin with a lemma that removes the standing Resetting Assumption.

Lemma 3: *For the reset control system in* (16) *utilizing FORE and* $L(s)$ *given in* (22)*, the set of reset times* $\mathcal{T}(x_0)$ *is unbounded and discrete for all positive* b*,* ω_n *and* $\zeta \in (0,1)$*. Moreover, the reset action is periodic with period* $\tau_i \equiv \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$.

Proof: To prove the theorem it suffices to show that the reset time interval is constant; i.e., $\tau_i = \tau \stackrel{\triangle}{=} \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ for all integer $i > 1$. Without loss of generality we start with an initial condition $x_o \in \mathcal{M}$; i.e., $\left[\omega_n^2 \; 0 \; 0\right] x_o = 0$. Again, without loss of generality we only consider x_o such that $\|x_{pco}\| = 1$, where $x_{pco} \in \mathbb{R}^2$ denotes the initial state of $P(s)C(s)$. Since $x_o \in \mathcal{M}$ we have $x_{pco} = \left[0 \; 1\right]'$ and therefore $x_o = \left[0 \; 1 \; x_{ro}\right]'$, where $x_{ro} \in \mathbb{R}$ is the initial state of the FORE. For τ_i to qualify as a reset time interval, the following condition must be satisfied:

$$\left[\omega_n^2 \; 0 \; 0\right] e^{A_{c\ell}\tau_i} A_R \begin{bmatrix} 0 \\ 1 \\ x_{ro} \end{bmatrix} = 0,$$

where $A_{c\ell} = \begin{bmatrix} A_{pc} & B_{pc} \\ -C_{pc} & -b \end{bmatrix}$ and $A_R = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$. This gives $\tau_i \equiv \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$. The next step in the proof is to show that τ is a valid reset time interval. To do this we need to show that $x_r(\tau) \neq 0$; i.e.,

$$\left[0 \; 0 \; 1\right] e^{A_{c\ell}\tau} A_R \begin{bmatrix} 0 \\ 1 \\ x_{ro} \end{bmatrix} \neq 0,$$

which is equivalent to saying

$$\frac{2\omega_n}{b^2 - 2\zeta\omega_n b + \omega_n^2} e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \left(1 - e^{-\frac{b}{\zeta\omega_n}}\right) \neq 0.$$

This is true for all positive b, ω_n and $\zeta \in (0,1)$ and hence concludes the proof.

Our next theorem shows that this class of reset control system is quadratically and BIBO stable and enjoys the previously described internal-model and superposition properties for all b, ζ and ω_n.

Theorem 4: *For the reset control system described in* (16) *utilizing FORE with* $P(s)C(s)$ *given in* (22)*, there exists a* $\beta \in \mathbb{R}$ *such that* $H_\beta(s)$ *in* (8) *is SPR for all positive* b*,* ζ *and* ω_n*. Consequently, such reset control systems are always quadratically stable, BIBO stable and enjoy the internal model and superposition properties of Theorem 2 and Corollary 1.*

Proof: First we note that if $\zeta \geq 1$, the reset control system becomes equivalent to its base-linear system, which is stable. Therefore we only need to consider the case when the system is underdamped; i.e., $\zeta < 1$. By Lemma 3, the Resetting Assumption is satisfied for reset control systems with $P(s)C(s)$ given in (22) for all positive values of b, ω_n and $\zeta \in (0,1)$. To show asymptotic stability we use Theorem 1 and show there exists a $\beta > 0$ such that

$$H_\beta(s) = \frac{s^2 + (2\zeta\omega_n + \beta)\, s + b\beta}{(s+b)\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

is SPR. The remainder of the proof deals with finding a $\beta \in \mathbb{R}$ such that $H_\beta(s)$ is SPR for all positive b, ζ and ω_n. We consider three cases:

Case 1: $(b > 2\zeta\omega_n)$ First, we form the partial fraction expansion

$$\begin{aligned} H_\beta(s) &= \frac{1}{\omega_n^2 - 2\zeta b\omega_n + b^2} \\ &\quad \cdot \left[\frac{b\,(b - 2\zeta\omega_n)}{s+b} + \frac{\omega_n^2 s + \left(\omega_n^2 - 2\zeta b\omega_n + b^2\right)\beta + \omega_n^2\,(2\zeta\omega_n - b)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right] \\ &\triangleq \frac{1}{\omega_n^2 - 2\zeta b\omega_n + b^2}\,[h_{11}(s) + h_{12}(s)]\,. \end{aligned}$$

Next, since $b > 2\zeta\omega_n$, then $\omega_n^2 - 2\zeta b\omega_n + b^2 > 0$. Hence, it suffices to show that both $h_{11}(s)$ and $h_{12}(s)$ are SPR; e.g., see [24]. Since $b\,(b - 2\zeta\omega_n) > 0$, then $h_{11}(s)$ is SPR. Finally, since $s^2 + 2\zeta\omega_n s + \omega_n^2$ is stable and the zero of $h_{12}(s)$ can be arbitrarily placed via β, there exists a β rendering $h_{12}(s)$ SPR.

Case 2: $(b = 2\zeta\omega_n)$ In this case it is clear that

$$H_\beta(s) = \frac{s + \beta}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is SPR for sufficiently small and positive β.

Case 3: ($b < 2\zeta\omega_n$) In this situation we write

$$H_\beta(s) = \frac{\frac{1}{s+(b-\beta)}}{1 + \frac{1}{s+(b-\beta)} \frac{[\omega_n^2 + 2\beta(\zeta\omega_n - b) + \beta^2]s + b(\beta^2 - b\beta + \omega_n^2)}{s^2 + (2\zeta\omega_n + \beta)s + b\beta}}$$
$$\triangleq \frac{h_{21}(s)}{1 + h_{21}(s)h_{22}(s)}.$$

Now it suffices to show that both $h_{21}(s)$ and $h_{22}(s)$ are SPR; again, see [24]. Clearly, $h_{21}(s)$ is SPR for any $\beta < b$ and a straightforward calculation shows that $h_{22}(s)$ is SPR for sufficiently small positive β. Hence, there exists a $\beta \in (0, b)$ such that $H_\beta(s)$ is SPR. This proves Case 3 and the theorem.

$H_\beta(s)$ is positive-real for all the examples in Section 2 In Section 2 we gave three examples motivating the use of reset control systems. We claimed that each was quadratically and BIBO stable and asymptotically tracked constant reference signals. This is certainly the case for the first two of these situations since Theorem 4 applies. To establish these properties for the third case we explicitly check the satisfaction of (8). Since $C_2(s)$ stabilizes, then $H_\beta(s)$ in (8) is asymptotically stable for all β. A simple search and computation shows that $\mathrm{Re}[H_\beta(j\omega)] > 0$ for all $\omega \geq 0$ when $\beta = 0.008$.

Overshoot, rise time and settling time In this section we analyze the reset control system (16) when $r(t) \equiv r_0$ and prove that the step-response maximum occurs during the time interval $(t_1, t_1 + \tau_0)$. The proof of the following can be found in [18].

Theorem 5: *Consider the reset control system described in* (16) *utilizing FORE with $L(s)$ given in* (22) *and $r(t) \equiv r_0$. Let $M_r \triangleq \sup_{t>0} |y(t) - r_0|$ denote the step-response maximum. Then,*

$$M_r = \max_{t \in [t_1, t_1 + \tau_0]} |y(t) - r_0|.$$

From Theorem 5 the step response maximum M_r is equal to the peak response in the first reset interval $[t_1, t_1 + \tau_0)$. In [3], this overshoot value has been explicitly computed in terms of b, ζ, and ω_n as repeated below:

$$M_r = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} - \Delta$$

where

$$\Delta = \begin{cases} \frac{R\left[4M^2\zeta^2 e^{-\zeta\mu} - 2\zeta M(1-4\zeta^2 M)e^{-\mu/\zeta M}\right]}{1-4\zeta^2 M + 4\zeta^2 M^2}; & \zeta \geq 0.5 \\ \frac{R\left[M^2 e^{-\zeta\mu} - M(1-2\zeta M)e^{-\mu/M}\right]}{1-2\zeta M + M^2}; & \zeta \leq 0.5 \end{cases},$$

$$R = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}} \cos^{-1}\zeta}; \quad M = \frac{\omega_c}{b}; \quad \mu = \frac{\pi - \cos^{-1}\zeta}{\sqrt{1-\zeta^2}}$$

and where ω_c is the unity-gain crossover frequency of $P(j\omega)C(j\omega)$.

Since the reset control system (16) behaves as a linear system before its first reset, then its rise time is that of its base-linear system ($\approx \frac{1.8}{\omega_n}$). The 2% settling time t_s can be computed using [18] adjacent intervals of y are shown to be scaled copies of each other. Indeed, using this, the settling time is computed as

$$t_s = \frac{k\pi}{\sqrt{1-\zeta^2}\omega_n}$$

where k is the smallest integer satisfying $|p_{11}(\tau_0)|^k M_r < 0.02$.

7 Conclusion

This paper has given a summary overview of reset control. It provided a number of motivating examples, both theoretical and experimental, and a framework for establishing basic feedback loop properties such as stability, steady-state and transient performance. One area of present study is the response of reset control systems to high-frequency sensor noise. Such results could help give a complete a description of classical properties of reset control systems.

A Proof of Proposition 1

We will first introduce some notation. Given an $r \in \mathbb{R}$, we define

$$\mathcal{B}_r \stackrel{\triangle}{=} \{\xi \in \Omega : \|\xi\| \leq r\}$$

and given a $\beta \in \mathbb{R}$ and $\mathcal{B}_r$ as above, we define

$$\Omega_\beta \stackrel{\triangle}{=} \{\xi \in \mathcal{B}_r : V(\xi) \leq \beta\}.$$

We will now show that the equilibrium point, $x = 0$, is stable. Given an $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ and let $\alpha = \min_{\|x\|=r} V(x)$. Since V is positive-definite we have $\alpha > 0$. We take $\beta \in (0, \alpha)$. Inequalities (4) and (5) imply that $V(x(t)) \leq V(x_o) \leq \beta$ for all t, and hence any trajectory starting in Ω_β will remain in Ω_β for all t.

Since $V(x)$ is continuous and $V(0) = 0$, there exists a $\delta > 0$ such that $V(x) < \beta$ for all $x \in \mathcal{B}_\delta$. Therefore, $\mathcal{B}_\delta \subset \Omega_\beta \subset \mathcal{B}_r$, and $x_o \in \mathcal{B}_\delta$ implies that $x(t) \in \mathcal{B}_r$ for all t. Therefore $\|x_o\| < \delta$ implies $\|x(t)\| < r \leq \varepsilon$ for all t, and hence $x = 0$ is a stable equilibrium point.

To show asymptotic stability we need to show that $x(t) \to 0$ as $t \to \infty$; that is, for every $a > 0$, there is a $T > 0$ such that $\|x(t)\| < a$ for all $t > T$. It suffices to show that $V(x(t)) \to 0$ as $t \to \infty$.

By (4) and (5), $V(x(t))$ is non-increasing. We now assume that it is decreasing when $x \in \Omega / \{0\}$ by (6), or when $x \in \Omega \cap \mathcal{M}$ by (7). Since $V(x)$ is bounded from below by zero, then there exists a $c \geq 0$ such that

$$V(x(t)) \to c \geq 0; \quad t \to \infty. \tag{23}$$

To show that $c = 0$, we proceed by contradiction and suppose $c > 0$. By continuity of $V(x)$, there exists a $d > 0$ such that $\mathcal{B}_d \subset \Omega_c$. The inequality (23) implies that the trajectory $x(t)$ lies outside the ball $\mathcal{B}_d$ for all t. We proceed with the proof in two cases.

Case 1: ((6) holds) Let $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$. Then, $\gamma > 0$ by (6) and because $V(x)$ is continuously differentiable . It follows that for $t_{i+1} \geq t > t_i$,

$$\begin{aligned} V(x(t)) &= V(x_o) + \sum_{n=1}^{i} \left[\int_{t_{n-1}}^{t_n} \dot{V}(x(\tau)) d\tau + \Delta V(x(t_n)) \right] + \int_{t_i}^{t} \dot{V}(x(\tau)) d\tau \\ &\leq V(x_o) - \gamma \sum_{n=1}^{i} \int_{t_{n-1}}^{t_n} d\tau + \gamma \int_{t_i}^{t} d\tau \\ &= V(x_o) - \gamma \left[\sum_{n=1}^{i} (t_n - t_{n-1}) + t - t_i \right] \\ &= V(x_o) - \gamma t. \end{aligned}$$

Since the right-hand side will eventually become negative, the inequality (23) contradicts the assumption that $c > 0$.

Case 2: ((7) holds) Now let $-\gamma = max_{d \leq \|x\| \leq r, x \in \mathcal{M}} \Delta V(x)$. Then, $\gamma > 0$ by (7) and because $V(x)$ is continuously differentiable. It follows that for $t_{i+1} \geq t > t_i$,

$$\begin{aligned} V(x(t)) &= V(x_o) + \sum_{n=1}^{i} \left[\int_{t_{n-1}}^{t_n} \dot{V}(x(\tau)) d\tau + \Delta V(x(t_n)) \right] + \int_{t_i}^{t} \dot{V}(x(\tau)) d\tau \\ &\leq V(x_o) - \sum_{n=1}^{i} \gamma \\ &= V(x_o) - i\gamma. \end{aligned}$$

Since the right-hand side will eventually become negative, the inequality (23) contradicts the assumption that $c > 0$. Therefore, $V(x(t)) \rightarrow 0$ as $t \rightarrow \infty$, and proof is completed.

B Proof of Proposition 2:

Given any point $q \in \mathbb{R}^n$, define $\beta = V(q) > 0$. Since $V(x)$ is radially unbounded, given any $\beta > 0$, there exists an $r > 0$ such that $V(x) > \beta$ for all $\|x\| > r$. Given r, we define $\mathcal{B}_r \triangleq \{\xi \in \Omega : \|\xi\| \leq r\}$ and given β and $\mathcal{B}_r$ as above we define $\Omega_\beta \triangleq \{\xi \in \mathcal{B}_r : V(\xi) \leq \beta\}$. The rest of the proof is similar to that of Proposition 1.

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