

Analysis of Reset Control Systems Consisting of a FORE and Second-Order Loop*

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Abstract

Reset controllers consist of two parts - a linear compensator and a reset element. The linear compensator is designed, in the usual ways, to meet all closed-loop performance specifications while relaxing the overshoot constraint. Then, the reset element is chosen to meet this remaining step-response specification. In this paper, we consider the case when such linear compensation results in a second-order (loop) transfer function and where a first-order reset element (FORE) is employed. We analyze the closed-loop reset control system addressing performance issues such as stability, steady-state response and transient performance.

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1 Introduction

This paper¹ is one in a sequence [1]-[5] describing recent research on reset control systems. While the original concept of reset elements and reset control design was introduced in the late 50's and early 70's; see [6] and [7], it's only now that we see broader interest and application ([8], [9], [10]) as well as analytical study ([11], [12]).

The main purpose for introducing reset elements is to mitigate some of the tradeoffs suffered by linear, time-invariant (LTI) feedback control systems. For instance, [13] presents control specifications that cannot be met using linear control but achievable using reset. Experimental work in [4] and [14] further confirm advantage while stability and asymptotic performance analysis is conducted in [1], [4] and [5]. The present paper continues this research by focusing on a special class of reset control system consisting of a first-order reset element and second-order loop transfer function.

Reset controllers, as introduced in [6] and further developed by [7] and [14], consist of two distinct parts – a linear compensator $C(s)$ and a reset element as shown in Figure 1. A reset element is simply a linear filter whose output is reset to zero when the filter input is zero. Special cases include the Clegg integrator [6] and the first-order reset element (FORE) [7] used in this paper. Commensurate to their structure, design of a reset controller proceeds in two steps. First, $C(s)$ is synthesized to meet control system specifications, with relaxed overshoot constraint, while the FORE is selected to meet this transient-response specification. The linear design can result in a loop transfer function $L(s) = P(s)C(s)$ dominated by a complex pole pair – a situation that this paper concentrates. Our work exploits this case and develops sharper results than made in [4] and [5]. For example, while these more general papers give sufficient conditions for asymptotic stability, our new results give a testable condition (see Section 3, Theorem 1) that is both necessary and sufficient. Similarly, we give a stronger BIBO stability condition in Theorem 2 of Section 4. Finally, the simpler setup considered here allows us to fully characterize the step response as summarized in Theorems 3 and 4 of Section 5. They also allow us to draw comparison to classical linear control systems described by second-order transfer functions $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ as discussed in Section 6. Before getting to these results, we first introduce the dynamics of reset control systems.

2 Dynamics of Reset Control

In this paper we focus on the reset control system in Figure 1 where the first-order reset element (FORE) is described by the impulsive differential equation [12]:

$$\begin{aligned} \dot{x}_f(t) &= -bx_f(t) + e(t); & e(t) &\neq 0 \\ x_f(t^+) &= 0; & e(t) &= 0 \end{aligned}$$

¹A preliminary version of this paper [3] was presented at the *American Control Conference*, Chicago, IL, 2000

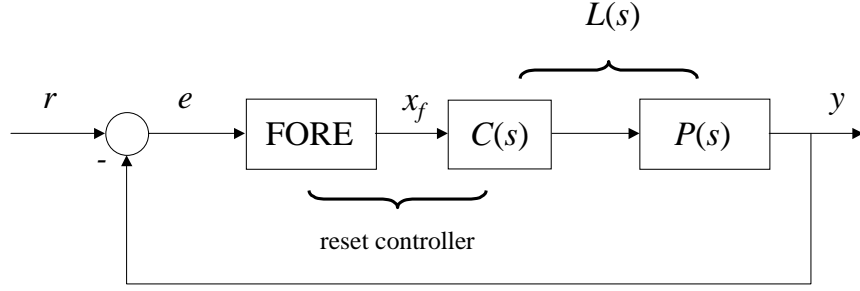


Figure 1: Block diagram of the reset control system considered in this paper. The reset controller consists of two parts: a linear compensator $C(s)$ and a FORE reset element.

where x_f is its state, e is the system error and b the FORE's pole; see [7]. To avoid degeneration to an LTI system, we assume that the FORE continually resets. We collect these reset times in the unbounded set

$$I = \{t_i \mid e(t_i) = 0, t_i > t_{i-1} + \sigma, \sigma > 0, i = 1, 2, \dots, \infty\}$$

where we assume that adjacent reset times are no closer than σ . This assumption is technically motivated by a desire to have closed-loop solutions continuable over $[0, \infty)$, but is met when FORE is digitally implemented and the sampling period is a lower bound to σ .

A state-space description of the reset control system is:

$$\begin{aligned} \dot{x}_\ell(t) &= Ax_\ell(t) + Bx_f(t) \\ \dot{x}_f(t) &= -Cx_\ell(t) - bx_f(t) + r(t); \quad t \notin I \\ x_f(t^+) &= 0; \quad t \in I \\ y(t) &= Cx_\ell(t) \end{aligned} \quad (1)$$

where $\{A, B, C\}$ denotes a minimal realization of $L(s)$ with state $x_\ell(t) \in \mathfrak{R}^n$. Let the closed-loop state be

$$x \triangleq \begin{bmatrix} x_f \\ x_\ell \end{bmatrix}.$$

Given $(x_f(0), x_\ell(0))$, the solution to (1) is piecewise left-continuous on the intervals $(t_i, t_{i+1}]$. In the absence of resetting, (1) reduces to the following linear system:

$$\dot{z}(t) \triangleq A_{cl}z(t) + \begin{bmatrix} 0 \\ r(t) \end{bmatrix}; \quad z(0) = x(0) \quad (2)$$

where

$$A_{cl} = \begin{bmatrix} A & B \\ -C & -b \end{bmatrix}.$$

In the sequel, we refer to (2) as the *base-linear system*. Furthermore, we will restrict our attention to second-order loop transfer functions so that $n = 2$, $x_\ell = [x_{\ell 1}, x_{\ell 2}]'$ and $x(t) \in \mathfrak{R}^3$. Finally, without loss of generality, we assume $C = [0 \ 1]$.

3 Asymptotic Stability

For asymptotic stability, we consider (1) with $r(t) \equiv 0$. Between successive reset times t_i and t_{i+1} , the closed-loop system behaves as the LTI system:

$$\dot{x}(t) = A_{cl}x(t), \quad t \in (t_i, t_{i+1}].$$

Therefore,

$$x(t) = e^{A_{cl}(t-t_i)}x(t_i^+), \quad t \in (t_i, t_{i+1}]. \quad (3)$$

By definition, the reset times t_i are characterized by $e(t_i) = 0$. Since $y(t) = x_{\ell 2}(t)$, then at each t_i we have $x_{\ell 2}(t_i) = 0$ and $x_f(t_i^+) = 0$. Therefore, (3) becomes

$$x(t) = \begin{bmatrix} p_{11}(t-t_i) \\ p_{21}(t-t_i) \\ p_{31}(t-t_i) \end{bmatrix} x_{\ell 1}(t_i), \quad t \in (t_i, t_{i+1}] \quad (4)$$

where $p_{ij}(t)$ denotes the (i, j) th entry of $e^{A_{cl}t}$, the state transition matrix of the base-linear system (2). Our first lemma characterizes some properties of (1).

Lemma 1: *Assume $r(t) \equiv 0$ and let $\tau_0 > 0$ denote the smallest number for which $p_{21}(\tau_0) = 0$. Then, (1) enjoys the following properties:*

1. $t_{i+1} - t_i = \tau_0$ for $i = 1, 2, \dots, \infty$.
2. $x(t + \tau_0) = p_{11}(\tau_0)x(t)$ for all $t \geq t_1$.

Proof: The reset time t_{i+1} is defined as the first time instant after t_i for which $x_2(t_{i+1}) = 0$. It follows from (4) that $x_2(t_{i+1}) = p_{21}(t_{i+1} - t_i)x_1(t_i) = 0$. The case $x_1(t_i) = 0$ is trivial. So, assume $x_1(t_i) \neq 0$. Therefore, $t_{i+1} - t_i$ is the smallest value such that $p_{21}(t_{i+1} - t_i) = 0$. Hence, $t_{i+1} - t_i = \tau_0$. This proves the first claim. From (4), $x_1(t_{i+1}) = p_{11}(\tau_0)x_1(t_i)$. Substituting this back into (4) gives $x(t + \tau_0) = p_{11}(\tau_0)x(t)$. \square

Lemma 1 describes an important feature of the trajectory of a second-order reset control system under zero input. Namely, the reset intervals $t_{i+1} - t_i$ are constant and the trajectories over adjacent intervals are simply copies scaled by the factor $p_{11}(\tau_0)$. This property will be illustrated in Section 6. The following main result is now obvious.

Theorem 1: *The reset control system (1) is asymptotically stable if and only if $|p_{11}(\tau_0)| < 1$.*

\square

4 BIBO Stability

This section develops a sufficient condition for bounded-input, bounded-output (BIBO) stability. The reset control system (1) is said to be BIBO stable if bounded² inputs r produce bounded outputs y . When $t \in (t_i, t_{i+1})$ the reset control system behaves as the LTI system (2) so that

$$x(t) = e^{A_{cl}(t-t_i)}x(t_i^+) + \int_{t_i}^t e^{A_{cl}(t-\sigma)} \begin{bmatrix} 0 \\ r(\sigma) \end{bmatrix} d\sigma.$$

Since $y(t) = x_{\ell 2}(t)$, then $x_{\ell 2}(t_i) = r(t_i)$. Hence,

$$\begin{aligned} x_{\ell 1}(t) &= p_{11}(t-t_i)x_{\ell 1}(t_i) + p_{12}(t-t_i)r(t_i) + \int_{t_i}^t p_{13}(t-\sigma)r(\sigma)d\sigma; \\ x_{\ell 2}(t) &= p_{21}(t-t_i)x_{\ell 1}(t_i) + p_{22}(t-t_i)r(t_i) + \int_{t_i}^t p_{23}(t-\sigma)r(\sigma)d\sigma; \end{aligned} \quad (5)$$

where again, $p_{ij}(t)$ is the (i, j) th entry of $e^{A_{cl}t}$. We have the following lemma:

Lemma 2: *Assume A_{cl} is asymptotically stable. If there exists an M such that $|x_{\ell 1}(t_i)| < M$ for all $i = 1, 2, \dots, \infty$, then (1) is BIBO stable.*

Proof: Since A_{cl} is stable, then from (5) there exist constants α and β such that $|x_{\ell 2}(t)| < \beta |x_{\ell 1}(t_i)| + \alpha$ for all $i = 1, 2, \dots, \infty$ and all $t \in (t_i, t_{i+1})$. It follows that y is bounded. \square

The main result of this section is as follows.

Theorem 2: *Assume A_{cl} is asymptotically stable. If there exists a $\gamma < 1$ such that $|p_{11}(\tau_i)| \leq \gamma$ for all $i = 1, 2, \dots, \infty$, then (1) is BIBO stable.*

Proof: From (5), we have

$$x_{\ell 1}(t) = p_{11}(\tau_i)x_{\ell 1}(t_i) + p_{12}(\tau_i)r(t_i) + \int_{t_i}^{\tau_i} p_{13}(\tau_i - \sigma)r(\sigma)d\sigma.$$

Because A_{cl} is asymptotically stable, there must exist a positive constant α such that

$$\begin{aligned} |x_{\ell 1}(t_{i+1})| &< |p_{11}(\tau_i)| |x_{\ell 1}(t_i)| + \alpha \\ &< \gamma |x_{\ell 1}(t_i)| + \alpha \\ &< \gamma^i |x_{\ell 1}(t_1)| + \frac{1 - \gamma^i}{1 - \gamma} \alpha \\ &< |x_{\ell 1}(t_1)| + \frac{1}{1 - \gamma} \alpha. \end{aligned}$$

So, there exists an M such that $|x_{\ell 1}(t_i)| < M$ for all $i = 1, 2, \dots, \infty$. From Lemma 2, (1) is BIBO stable. \square

²A signal z is said to be bounded if there exists a constant M such that $|z(t)| \leq M$ for all $t > 0$.

Remark: In Section 6 we focus on a particular class of second-order $L(s)$, see (7) and show in the Appendix that the conditions of Theorem 2 *are always satisfied*. Thus, Theorem 2 is not vacuous.

5 Properties of the Step Response

In this section we analyze the response of reset control systems (1) to a constant reference signal $r(t) \equiv r_0$.

5.1 Steady-state response

Assume that $L(s)$ contains at least one integrator. Consequently, there exists an $\xi \in \mathfrak{R}^3$ such that $A\xi = 0$ and $C\xi = r_0$. Define the state transformation $\tilde{x}_\ell(t) = x_\ell(t) - \xi$ and associated transformed system:

$$\begin{aligned} \dot{\tilde{x}}_\ell(t) &= A\tilde{x}_\ell(t) + B\tilde{x}_f(t) \\ \dot{\tilde{x}}_f(t) &= -C\tilde{x}_\ell(t) - b\tilde{x}_f(t); & t \notin I \\ \tilde{x}_f(t^+) &= 0; & t \in I \\ \tilde{y} &= C\tilde{x}_\ell \end{aligned} \tag{6}$$

where

$$I = \{t_i \mid \tilde{y}(t_i) = 0, t_i > t_{i-1} + \sigma, \sigma > 0, i = 1, 2, \dots, \infty\}.$$

The following is straightforward.

Lemma 3: *If $L(s)$ has at least one integrator, then system (1) and system (6) are equivalent under the state transformation $\tilde{x}_\ell(t) = x_\ell(t) - \xi$. \square*

This lemma states that we need only analyze the zero-input reset control system (6). Indeed, since

$$y(t) = \tilde{y}(t) + C\xi = \tilde{y}(t) + r_0$$

the response of a second-order reset control system (1) to $r(t) \equiv r_0$ is equal to its zero-input response plus r_0 . The following is immediate from the results in Section 3.

Lemma 4: *Assume $L(s)$ contains one integrator and $r(t) \equiv r_0$. Then the following are true for (6):*

1. *Let $\tau_0 > 0$ be the smallest number satisfying $p_{21}(\tau_0) = 0$. Then, $t_{i+1} = t_i + \tau_0$ for $i = 1, 2, \dots, \infty$.*
2. *The equilibrium state is asymptotically stable if and only if $|p_{11}(\tau_0)| < 1$.*
3. *The output y satisfies $y(t + \tau_0) - r_0 = p_{11}(\tau_0)[y(t) - r_0]$ for all $t \geq t_1$. \square*

Asymptotic tracking of step inputs now follows directly from the last two claims made in Lemma 4.

Theorem 3: *Assume $L(s)$ contains at least one integrator and (1) is asymptotically stable. Let $r(t) \equiv r_0$. Then, $\lim_{t \rightarrow \infty} y(t) = r_0$. \square*

5.2 Transient response

Our next result proves that the step-response maximum of (1) occurs during the time interval $(t_1, t_1 + \tau_0)$. This proves to be valuable since, as we see in the next section, this maximum can be related to the FORE's pole b when $L(s)$ takes a standard second-order form.

Theorem 4: *Assume $L(s)$ contains at least one integrator, (1) is asymptotically stable³ and $r(t) \equiv r_0$. Let $M_r \triangleq \sup_{t > 0} |y(t) - r_0|$ denote the step-response maximum. Then,*

$$M_r = \max_{t \in [t_1, t_1 + \tau_0]} |y(t) - r_0|.$$

Proof: From Lemma 4, we have

$$y(t + \tau_0) - r_0 = p_{11}(\tau_0)[y(t) - r_0]$$

for all $t \geq t_1$. Since system (1) is asymptotically stable, then, from Lemma 4, $|p_{11}(\tau_0)| < 1$. Hence,

$$\sup_{t > t_1 + \tau_0} |y(t) - r_0| < \max_{t \in [t_1, t_1 + \tau_0]} |y(t) - r_0|.$$

It then follows that

$$\sup_{t > 0} |y(t) - r_0| = \max_{t \in [t_1, t_1 + \tau_0]} |y(t) - r_0|.$$

\square

Since the reset control system (1) behaves as a linear system before its first reset, then its rise time is that of its base-linear system. The 2% settling time t_s can be computed using the third statement in Lemma 4 where outputs over adjacent intervals are shown to be scaled copies of each other. Indeed, using this, the settling time is computed as $t_s = k\tau_0$ where k is the smallest integer satisfying the inequality $|p_{11}(\tau_0)|^k M_r < 0.02$. We now illustrate these properties for a special class of second-order reset control systems.

³Recall from Theorem 1 that (1) is asymptotically stable iff $|p_{11}(\tau_0)| < 1$.

6 Reset Control Systems with $L(s) = \frac{\omega_n^2(s+b)}{s(s+2\zeta\omega_n)}$

In this section we illustrate the results in Section 5 on a class of second-order reset control systems where

$$L(s) = \frac{\omega_n^2(s+b)}{s(s+2\zeta\omega_n)}; \quad b, \zeta, \omega_n > 0. \quad (7)$$

A reason for considering this $L(s)$ is that the base-linear system of (1) has the transfer function⁴

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

which is descriptive of linear feedback systems dominated by a complex pole pair. For this class of reset control system, the corresponding A_{cl} is asymptotically stable. Moreover, in the Appendix we show that $|p_{11}(\tau)| < 1$ for all positive parameters (b, ζ, ω_n) and any $\tau > 0$. Consequently, Theorems 1 and 2 are in effect and this class of reset control system is asymptotically and BIBO stable. As far as the step response is concerned, we can invoke Theorem 3 and conclude that the response asymptotically tracks a constant reference. From Theorem 4 the step response maximum M_r is equal to the peak response in the first reset interval $[t_1, t_1 + \tau_0)$. In [7], this overshoot value has been explicitly computed in terms of b, ζ , and ω_n as repeated below:

$$M_r = e^{-\pi\zeta/\sqrt{1-\zeta^2}} - \Delta \quad (8)$$

where

$$\Delta = \begin{cases} \frac{R[4M^2\zeta^2e^{-\zeta\mu} - 2\zeta M(1-4\zeta^2M)e^{-\mu/\zeta M}]}{1-4\zeta^2M+4\zeta^2M^2}; & \zeta \geq 0.5 \\ \frac{R[M^2e^{-\zeta\mu} - M(1-2\zeta M)e^{-\mu/M}]}{1-2\zeta M+M^2}; & \zeta \leq 0.5 \end{cases}$$

$$R = e^{\frac{-\zeta}{\sqrt{1-\zeta^2}} \arccos \zeta}; \quad M = \frac{\omega_c}{b}; \quad \mu = \frac{\pi - \arccos \zeta}{\sqrt{1-\zeta^2}}$$

and where ω_c is the unity-gain crossover frequency of $|L(j\omega)|$. The rise time is exactly that of the base-linear system ($\approx \frac{1.8}{\omega_n}$) with the settling time given by

$$t_s = \frac{k\pi}{\sqrt{1-\zeta^2}\omega_n}$$

where k is the smallest integer satisfying $|p_{11}(\tau_0)|^k M_r < 0.02$.

To further illustrate, consider:

$$L(s) = \frac{s+1}{s(s+0.2)}$$

⁴Notice that the zero term $s+b$ is included in $L(s)$ to stably cancel the corresponding pole term in the FORE when no reset occurs.

corresponding to the choices: $b = 1$, $\zeta = 0.1$ and $\omega_n = 1$. The step response of the reset control system (1) and its base-linear system (2) are compared in Figure 2. The value of $b = 1$ is chosen in accordance with (8) to reduce the overshoot in the reset control system to approximately 40% as compared to almost 70% in the base-linear system's response. The settling time is smaller while the rise times are similar. This simple comparison shows some of potential of using reset control to improve the tradeoffs in feedback control systems. The interested reader is directed to [4], [5], [13] and [14], for further discussion and illustration of reset control.

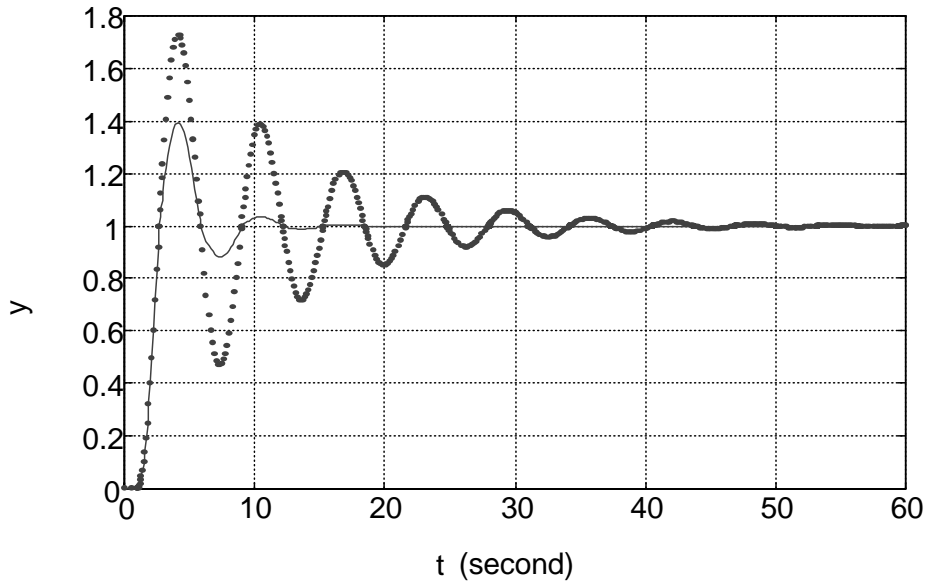


Figure 2: Comparison of step responses for reset control system (solid) and its base-linear system (dotted).

7 Conclusion

In this paper, we have focused on reset control systems comprised of a FORE reset element and a second-order loop. We gave sharp results for asymptotic and BIBO stability, asymptotic tracking of constant inputs and transient-response properties such as rise time, overshoot and settling time.

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Appendix

A Showing $|p_{11}(\tau)| < 1$

This Appendix shows that $|p_{11}(\tau)| < 1$ for all positive (b, ζ, ω_n) and all $\tau > 0$, where

$$p_{11}(\tau) = \frac{(c^2 + d^2)[e^{-b\tau} + \frac{b}{d}e^{-c\tau} \sin(d\tau)] + (b^2 - 2bc)e^{-c\tau}[\cos(d\tau) + \frac{c}{d} \sin(d\tau)]}{c^2 + d^2 + b^2 - 2bc} \quad (9)$$

is the (1, 1) element of $e^{A_{cl}\tau}$ and where $c = \zeta\omega_n$ and $d = \sqrt{1 - \zeta^2}\omega_n$.

Case 1: ($b \leq c$) First,

$$\begin{aligned} \frac{dp_{11}(\tau)}{d\tau} &= \frac{(c^2 + d^2)b\{-e^{-b\tau} + e^{-c\tau}[\cos(d\tau) - \frac{(b-c)}{d} \sin(d\tau)]\}}{c^2 + d^2 + b^2 - 2bc} \\ &= \frac{(c^2 + d^2)be^{-b\tau}\{e^{(b-c)\tau}[\cos(d\tau) + \frac{(c-b)}{d} \sin(d\tau)] - 1\}}{b^2 - 2bc + c^2 + d^2}. \end{aligned} \quad (10)$$

Since $c - b \geq 0$, then

$$\begin{aligned} e^{(b-c)\tau}[\cos(d\tau) + \frac{(c-b)}{d} \sin(d\tau)] &\leq e^{(b-c)\tau} \left[1 + (c-b)\tau \left| \frac{\sin(d\tau)}{d\tau} \right| \right] \\ &\leq \frac{1 + (c-b)\tau}{e^{(c-b)\tau}} \\ &\leq 1. \end{aligned}$$

From (10), it is easy to prove that $\frac{dp_{11}(\tau)}{d\tau} \leq 0$. From (9), $p_{11}(0) = 1$. Hence, $|p_{11}(\tau)| < 1$, for all $\tau > 0$.

Case 2: ($b > c$) Formally setting the numerator of (10) to zero we obtain

$$\frac{(b-c)}{d} \sin(d\tau_m) = \cos(d\tau_m) - e^{(c-b)\tau_m}. \quad (11)$$

Then,

$$\max_{0 \leq \tau < \infty} [p_{11}(\tau)] = \max\{p_{11}(0), p_{11}(\infty), p_{11}(\tau_m)\}. \quad (12)$$

Substituting (11) into (9) gives

$$\begin{aligned} p_{11}(\tau_m) &= \frac{-ce^{-b\tau_m} + be^{-c\tau_m} \cos(d\tau_m)}{b-c} \\ &\leq \frac{-ce^{-b\tau_m} + be^{-c\tau_m}}{b-c}. \end{aligned}$$

Let

$$g(t) \triangleq \frac{-ce^{-bt} + be^{-ct}}{b-c}$$

with $g(0) = 1$ and where

$$\frac{dg(t)}{dt} = \frac{-bc(e^{-ct} - e^{-bt})}{(b - c)}.$$

Because $b > c$, then $\frac{dg(t)}{dt} < 0$ for all $t > 0$. It then follows that $g(t) < g(0) = 1$ for all $t > 0$ since $g(\infty) = 0$. Therefore, $p_{11}(\tau_m) \leq g(\tau_m) < 1$, $p_{11}(0) = 1$ and $p_{11}(\infty) = 0$. Thus, from (12), $|p_{11}(\tau)| < 1$ for all $\tau > 0$. Done.