On Reset Control Systems with Second-Order Plants [1]

Q. Chen‡, C.V. Hollot†, Y. Chait‡ and O. Beker†

‡MIE Department, †ECE Department, University of Massachusetts, Amherst, MA 01003

Abstract

Reset control has the potential of providing better trade-offs among competing specifications compared to LTI control. In this paper we consider a specific class of reset control systems consisting of a feedback interconnection between a linear second-order system and a so-called first-order reset element (FORE). Despite the simplicity of this feedback system, few theoretical results are available to quantify stability and performance. This paper develops a necessary and sufficient condition for asymptotic stability and a sufficient condition for BIBO stability. We also characterize steady-state response, overshoot, rise time and settling time to step input.

1 Introduction

It is well-known that there exist performance trade-offs in the design of linear feedback control system due to the constraints imposed by Bode's gain-phase relationship [10]. Specifically, high-frequency loop gain is limited by low-frequency specifications and stability margins. This phenomenon is called "cost of feedback" for which LTI design can not remedy. Reset control was introduced to improve this problem.

The basic idea of reset control is to reset the state of a linear controller to zero whenever its input meets a threshold. Typical reset controllers include the so-called Clegg-integrator [1] and first-order reset element (FORE) [2]. The former is a linear integrator whose output resets to zero when its input cross zero. The latter generalizes the Clegg concept to a lag filter $1/(s+b)$. The structure of reset control system under consideration is shown in Figure 1 where signals $r(t)$, $y(t)$, $e(t)$, $n(t)$ and $d(t)$ represent reference input, output, error, sensor noise and disturbance respectively. In the absence of resetting, the FORE behaves as the linear filter $1/(s+b)$. In this case, we refer to the resulting linear, closed loop system as the *base linear system*.

Reset control has the potential to provide improved

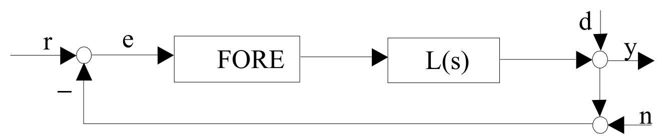

Figure 1: Block diagram of reset control system

performance tradeoffs in feedback control as exemplified by the work in [2]-[4]. This potential was also confirmed experimentally in [4] and [5]. To further illustrate, consider an example with $L(s) = (s+1)/s(s+0.2)$ and $b = 1$. Figure 2 shows that this reset control system achieves a 40% reduction in step response overshoot compared to its base linear system. Figure 3 shows that their responses to 2 rad/sec sinusoidal sensor noise are similar. The implication is that FORE provides better performance tradeoffs.

Despite its potential benefit, there is a lack of theoretical results for stability and performance of reset control system. To address this void, we continue along the line of research conducted in [7] and [8] and perform a detailed investigation of the stability and performance of reset control systems having second-order linear plants. First, we develop a necessary and sufficient condition for asymptotic stability and a sufficient condition for BIBO stability. These results are proven efficient and easy to check when applied to a specific class of reset control systems. Secondly we study the step response and characterize steady-state response, overshoot, rise time and settling time. For a specific class of reset control systems, we explicitly compute the maximum overshoot, rise time and settling time for the step response.

Finally, we want to point out that reset control action resembles a number of popular nonlinear control strategies including relay control , sliding mode control (SMC) and switching control. A common feature to all of these is the use of a switching surface to trigger change in the control signal. In relay control, SMC and switching control, the control law is defined differently on each side of the switching surface. In contrast, the same control law is used on both sides of the switching surface in reset control. A change takes place on

[1]Supported by the NSF under Grant No. CMS-9800612.
Email: qchen@ecs.umass.edu

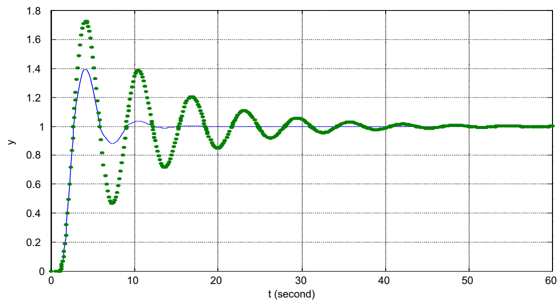

Figure 2: Comparison of step responses for reset control system (solid) and its base linear system (dot).

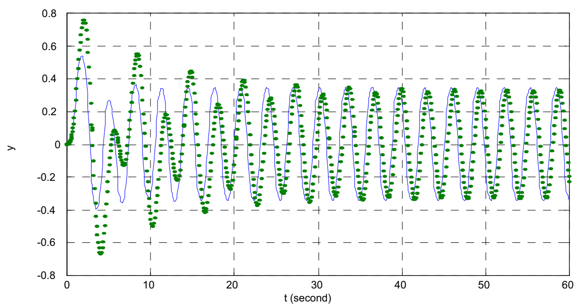

Figure 3: Comparison of responses to 2 rad/sec sinusoidal sensor noise for the reset control system (solid) and its base linear system (dot).

a fixed surface wherein the controller states are reset to zero. The reset action can be modeled as the injection of judiciously-timed, state-dependent impulses into an otherwise linear feedback system. This analogy is evident in the paper where we model reset control systems by impulsive differential equations; e.g., see [9]. However, the results for impulse differential equations are too conservative for our specific class of reset control systems.

2 State-Space Description

The structure of the reset control system under consideration is shown in the Figure 1. For simplicity, we only consider response to $r(t)$ and assume that $n(t)$ and $d(t)$ are zero. Response to either $n(t)$ or $d(t)$ can be dealt with in a similar way. The FORE element is described by the impulsive differential equation [9]:

$$\begin{array}{rcll} \dot{x}_c(t) & = & -bx_c(t) + e(t); & e(t) \neq 0 \\ x_c(t^+) & = & 0; & e(t) = 0 \end{array} \qquad (1)$$

where $x_c(t)$ is the state of FORE. The time instants when $e(t) = 0$ are called *reset times* and the set of reset times I is defined as

$$I = \{t_i \mid e(t_i) = 0, t_i > t_{i-1}, i = 1, 2, ...\}.$$

Assume that $\{A, B, C\}$ is a minimal realization of $L(s)$ and $x_p(t) \in \Re^n$ is the plant states. Then the state-space description of the reset control system in Figure 1 is

$$\begin{array}{rcll} \dot{x}_p(t) & = & Ax_p(t) + Bx_c(t) & \\ \dot{x}_c(t) & = & -Cx_p(t) - bx_c(t) + r(t); & t \notin I \\ x_c(t^+) & = & 0; & t \in I \\ I & = & \{t_i \mid Cx_p(t_i) - r(t_i) = 0,\ t_i > t_{i-1}\}. & \end{array} \qquad (2)$$

The output is $y(t) = Cx_p(t)$. Now, define $x(t) = [x_p^T(t)\ x_c(t)]^T$ and $A_{cl} = \begin{bmatrix} A & B \\ -C & -b \end{bmatrix}$. Then, when $t \in (t_i,\ t_{i+1}]$, the reset control system (2) behaves as the base linear system

$$\dot{x}\ (t) = A_{cl}x(t) + \begin{bmatrix} 0 \\ r(t) \end{bmatrix}, \qquad (3)$$

3 A Preliminary Result

When the input of (2), $r(t)$, is equal to a constant r_0, we can transform it into a new reset control system with zero input. First, we have the following lemma (see [11]).

Lemma 1 *If $L(s)$ has at least one integrator, then there exists a $x_0 \in \Re^n$ such that $Ax_0 = 0$, and $Cx_0 = r_0$.* ■

Define the state transformation $x_{p0}(t) = x_p(t) - x$ and associated transformed system:

$$\begin{array}{rcll} \dot{x}_{p0}(t) & = & Ax_{p0}(t) + Bx_c(t) & \\ \dot{x}_c(t) & = & -Cx_{p0}(t) - bx_c(t); & t \notin I \\ x_c(t^+) & = & 0; & t \in I \\ I & = & \{t_i \mid Cx_p(t_i) - \rho(t_i) = 0, t_i > t_{i-1}\}. & \end{array} \qquad (4)$$

We can prove the following theorem (see [11]).

Theorem 2 *If $L(s)$ has at least one integrator, then system (2) and system (4) are equivalent under the state transformation $x_{p0}(t) = x_p(t) - x_0$.* ■

Theorem 2 states that when the input $r(t)$ is constant, we only need to consider a reset control system (4) under zero input. This result holds for any reset control system provided that $L(s)$ has at least one integrator.

4 Response Under Zero Input

From now on, we will focus on the reset control systems (2) with a second-order plant $L(s)$ which we call a second-order reset control system. In this case, the plant state $x_p(t) = [x_1(t)\ x_2(t)]^T$. Then $x(t) = [x_1(t)\ x_2(t)\ x_c(t)]^T$. Without loss of generality we assume that $C = [0, 1]$. Therefore $y(t) = x_2(t)$.

In this section, we consider the zero-input case for which (2) becomes:

$$\begin{aligned} \dot{x}_p(t) &= Ax_p(t) + Bx_c(t) \\ \dot{x}_c(t) &= -Cx_p(t) - bx_c(t); \quad t \notin I \\ x_c(t^+) &= 0; \qquad\qquad\qquad\quad t \in I \end{aligned} \tag{5}$$

and the output is $y(t) = x_2(t)$. From (3), we know that between successive reset times t_i and t_{i+1}, the closed-loop system behaves as the LTI system:

$$\dot{x}\ (t) = A_{cl}x(t),\ t \in (t_i,\ t_{i+1}]$$

Therefore,

$$x(t) = e^{A_{cl}(t-t_i)}x(t_i^+),\ t \in (t_i, t_{i+1}]. \tag{6}$$

By definition, the reset times t_i are characterized by $e(t_i) = 0$. Since $y(t) = x_2(t)$ and $r(t) = 0$, at each t_i we have $x_2(t_i) = 0$ and $x_c(t_i^+) = 0$. Therefore, (6) becomes

$$x(t) = \begin{bmatrix} p_{11}(t - t_i) \\ p_{21}(t - t_i) \\ p_{31}(t - t_i) \end{bmatrix} x_1(t_i),\ t \in (t_i, t_{i+1}]. \tag{7}$$

where $p_{ij}(t)$ denotes the (i, j)th entry of $e^{A_{cl}t}$. We have following results:

Lemma 3 *Let $\tau_0 > 0$ denote the smallest number for which $p_{21}(\tau_0) = 0$. Then, the reset times t_i for system (5) satisfy $t_{i+1} - t_i = \tau_0$ for all i.*

Proof: The reset time t_{i+1} is defined as the first time instant after t_i for which $x_2(t_{i+1}) = 0$. It follows from (7) that $x_2(t_{i+1}) = p_{21}(t_{i+1} - t_i)x_1(t_i) = 0$. The case $x_1(t_i) = 0$ is trivial since $x(t)$ will stay at 0 after t_i. So, assume $x_1(t_i) \neq 0$. Therefore, $t_{i+1} - t_i$ is the smallest value such that $p_{21}(t_{i+1} - t_i) = 0$. Hence, $t_{i+1} - t_i = \tau_0$. Proof is completed. ∎

Theorem 4 *Assume $x(t)$ is a solution of system (5), then $x(t + \tau_0) = p_{11}(\tau_0)x(t)$ for any $t \geqslant t_1$.*

Proof: From Lemma 3, $t_{i+1} - t_i = \tau_0$. So, from (7) we have

$$x_1(t_{i+1}) = p_{11}(\tau_0)x_1(t_i). \tag{8}$$

Substitute back to (7), it is easy to get

$$x(t + \tau_0) = p_{11}(\tau_0)x(t) \tag{9}$$

Proof is completed. ∎

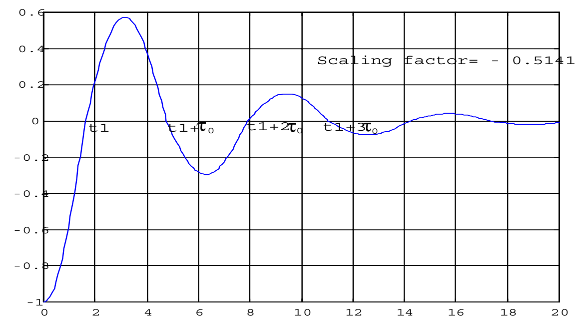

Figure 4: Response of Second-order Reset System under Zero Input

Corollary 5 *The output of system* (5) *satisfies $y(t + \tau_0) = p_{11}(\tau_0)y(t)$ for any $t \geqslant t_1$.* ∎

Theorem 4 describes an important feature of the trajectory $x(t)$ of second-order reset control system under zero input. When $t \in (t_{i+1}, t_{i+2}]$, $x(t) = [p_{11}(\tau_0)]^i x(t - i\tau_0)$ where $t - i\tau_0 \in (t_1, t_1 + \tau_0]$. In other words, the trajectory $x(t)$ after $t > t_1 + \tau_0$ is simply a copy of the trajectory of $x(t)$ over $(t_1, t_1 + \tau_0]$ scaled by factor $p_{11}(\tau_0)$. This feature is shown in Figure 4. The following result follows immediately.

Theorem 6 *The reset control system (5) is asymptotically stable if and only if $|p_{11}(\tau_0)| < 1$.* ∎

5 Step Response

In this section we make a further assumption that $L(s)$ has at least one integrator. In the case of step input, (2) becomes:

$$\begin{aligned} \dot{x}_p(t) &= Ax_p(t) + Bx_c(t) \\ \dot{x}_c(t) &= -Cx_p(t) - bx_c(t) + 1; \quad t \notin I \\ x_c(t^+) &= 0; \qquad\qquad\qquad\qquad\quad t \in I \\ y(t) &= Cx_p(t). \end{aligned} \tag{10}$$

From Theorem 2, it suffices to consider the following system:

$$\begin{aligned} \dot{x}_{p0}(t) &= Ax_{p0}(t) + Bx_c(t) \\ \dot{x}_c(t) &= -Cx_{p0}(t) - bx_c(t); \quad t \notin I \\ x_c(t^+) &= 0; \qquad\qquad\qquad\quad\ t \in I \\ y_0(t) &= Cx_{p0}(t). \end{aligned} \tag{11}$$

where $x_{p0}(t) = x_p(t) - x_0$, x_0 is from Lemma 1. Therefore,

$$y(t) = y_0(t) + Cx_0 = y_0(t) + 1.$$

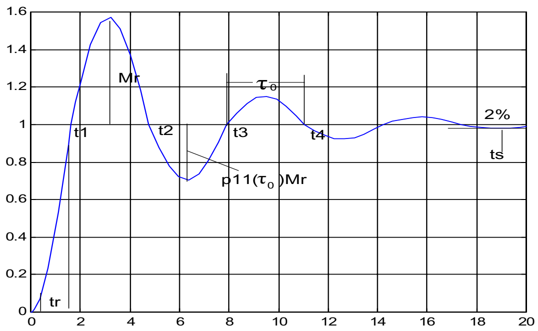

Figure 5: Step response of second-order reset system

The conclusion is that the step response of a second-order reset control system is equal to its response under zero input plus 1. This property is verified in comparing Figures 4 and 5. Therefore, the following results follows directly from the results in Section 4.

Corollary 7 *For system* (10) *let* $\tau_0 > 0$ *be the smallest number satisfying* $p_{21}(\tau_0) = 0$. *Then,* $t_{i+1} = t_i + \tau_0$ *for all* i. ∎

Corollary 8 *The equilibrium state of system* (10) *is asymptotically stable if and only if* $|p_{11}(\tau_0)| < 1$. ∎

Corollary 9 *The output* $y(t)$ *of system* (10) *satisfies* $y(t+\tau_0) - 1 = p_{11}(\tau_0)[y(t) - 1]$ *for any* $t \geqslant t_1$. ∎

The following are some characteristics of the step response of second-order reset control system (10). These results can be derived easily from Corollary 7-9.

Theorem 10 *If system* (10) *is asymptotically stable, then its output* $y(t)$ *satisfies* $\lim_{t\to\infty} y(t) = 1$. ∎

Theorem 11 *Let* $M_r = \max_{t>t_1} |y(t) - 1|$ *denotes the maximum overshoot of system* (10). *If system* (10) *is asymptotically stable, then* $M_r = \max_{t\in[t_1, t_1+\tau_0)} |y(t) - 1|$.

∎

Since the reset control system (10) behaves as a linear system before the first reset time, its rise time t_r will be the same as that of its base linear system.

The 2% settling time t_s can be computed using Corollary 9; namely, that the peak response in the reset interval $[t_i, t_i + \tau_0)$ is the peak response in the first reset interval $[t_1, t_1 + \tau_0)$, which is equal to M_r, scaled by $p_{11}(\tau_0)^k$ (see Figure 5). Thus, the settling time will be $t_s = k\tau_0$ where k is determined by the inequality $|p_{11}(\tau_0)|^k M_r < 0.02$.

6 BIBO Stability

This section develops a sufficient condition for BIBO stability. We assume that A_{cl} is stable and there exists a constant M such that $|r(t)| < M$ for all t.

When $t \in (t_i,\ t_{i+1})$, the reset system behaves as the LTI system (3) so that

$$x(t) = e^{A_{cl}(t-t_i)} x(t_i^+) + \int_{t_i}^{t} e^{A_{cl}(t-\sigma)} \begin{bmatrix} \mathbf{0} \\ r(\sigma) \end{bmatrix} d\sigma.$$

Since $y(t) = x_2(t)$, then $x_2(t_i) = r(t_i)$. Hence,

$$\begin{aligned} x_1(t) &= p_{11}(t-t_i)x_1(t_i) + p_{12}(t-t_i)r(t_i) \\ &\quad + \int_{t_i}^{t} p_{13}(t-\sigma)r(\sigma)d\sigma; \qquad (12) \\ x_2(t) &= p_{21}(t-t_i)x_1(t_i) + p_{22}(t-t_i)r(t_i) \\ &\quad + \int_{t_i}^{t} p_{23}(t-\sigma)r(\sigma)d\sigma; \qquad (13) \end{aligned}$$

where $p_{ij}(t)$ are the (i,j)th entry of $e^{A_{cl}t}$. We have the following lemma:

Lemma 12 *If* $x_1(t_i)$ *is bounded for all* t_i, *then* $y(t)$ *is bounded, i.e, there exists a constant* M_1 *such that* $|y(t)| < M_1$ *for all* t.

Proof: Since A_{cl} is stable, from (13) there must exist constants α and β such that

$$|x_2(t)| < \beta\, |x_1(t_i)| + \alpha M.$$

If $x_1(t_i)$ is bounded for all t_i, then $x_2(t)$ is bounded for all t. It follows that $y(t)$ is bounded. ∎

The main result of this section is as follows:

Theorem 13 *If there exists a* $\gamma < 1$ *such that* $|p_{11}(\tau_i)| \leqslant \gamma$ *for all reset intervals* $\tau_i = t_{i+1} - t_i$, *then* $y(t)$ *is bounded.*

Proof: From (12), we have

$$\begin{aligned} x_1(t) &= p_{11}(\tau_i)x_1(t_i) + p_{12}(\tau_i)r(t_i) \\ &\quad + \int_{t_i}^{t_{i+1}} p_{13}(t_{i+1}-\sigma)r(\sigma)d\sigma. \end{aligned}$$

Because A_{cl} is stable, there must exists a positive constant α_1 such that

$$\begin{aligned}
|x_1(t_{i+1})| &< |p_{11}(\tau_i)|\,|x_1(t_i)| + \alpha_1 M \\
&< \gamma\,|x_1(t_i)| + \alpha_1 M \\
&< \gamma^i\,|x_1(t_1)| + \frac{1-\gamma^i}{1-\gamma}\alpha_1 M \\
&< |x_1(t_1)| + \frac{1}{1-\gamma}\alpha_1 M.
\end{aligned}$$

So, $x_1(t_i)$ is bounded for all t_i. From Lemma 12 $y(t)$ is bounded. The proof is completed. ■

7 Specialized Linear Plant

In this section we specialize the results in stability and step response to a class of second-order reset control systems in which the linear plant is

$$L(s) = \frac{\omega_n^2(s+b)}{s(s+2\zeta\omega_n)} \tag{14}$$

where the parameters (b, ζ, ω_n) are positive. The reason for considering (14) is that its base linear system has the standard second-order transfer function $T(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$.

7.1 Asymptotic and BIBO Stability

For this class of reset control system, the corresponding A_{cl} is asymptotically stable. Moreover, we can show that that $|p_{11}(\tau)| < 1$ for all positive parameters (b, ζ, ω_n) and any $\tau > 0$. Consequently, from Theorem 6 and 13, this class of reset control system (14) is asymptotically and BIBO stable.

7.2 Step Response

We characterize the step response of the reset control system (14) as follows: First, we invoke Theorem 10 and conclude that the step response will asymptotically track the reference. Secondly, using Theorem 11, the maximum value of overshoot M_r is equal to the peak response in the first reset interval $[t_1, t_1 + \tau_0)$. This value has been given in [2]. Therefore the exact value of maximum overshoot M_r is

$$M_r = \exp\left[-\pi\zeta/\sqrt{1-\zeta^2}\right] - \Delta \tag{15}$$

where

$$\Delta = \begin{cases} \frac{R\left[4M^2\zeta^2 e^{-\zeta\mu} - 2\zeta M(1-4\zeta^2 M)e^{-\mu/\zeta M}\right]}{1-4\zeta^2 M + 4\zeta^2 M^2}; & \zeta \geq 0.5 \\ \frac{R\left[M^2 e^{-\zeta\mu} - M(1-2\zeta M)e^{-\mu/M}\right]}{1-2\zeta M + M^2}; & \zeta \leq 0.5 \end{cases}$$

$$R = \exp\left(\frac{-\zeta}{\sqrt{1-\zeta^2}}\arccos\zeta\right);\, M = \frac{\omega_c}{b};\, \mu = \frac{\pi - \arccos\zeta}{\sqrt{1-\zeta^2}}$$

ω_c is the crossover frequency of $L(s)$. Thirdly, the rise time t_r is exactly that of the base linear system $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$. Finally the settling time t_s is

$$t_s = k\pi/(\sqrt{1-\zeta^2}\omega_n) \tag{16}$$

where k satisfies $|p_{11}(\tau_0)|^k M_r < 0.02$.

8 Conclusion

In this paper, the stability and performance of second-order reset control system are investigated in detail. These results will constitute a good base for the further investigation and application of reset system.

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