ON THE CONVEXITY OF QFT BOUNDS AND ITS RELATION TO AUTOMATIC LOOP-SHAPING

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ABSTRACT

A difficult problem in QFT is the design of a nominal-loop function. A recent, promising direction for QFT loop-shaping involves convex optimization. However, the underlying QFT bounds are often non-convex sets requiring the bounds be approximated by convex sets to allow for convex optimization. In this paper, we propose a novice automatic loop-shaping technique via linear programming. Specifically, we transform the open-loop QFT bounds into closed-loop QFT bounds to reduce design conservation due to approximation. We also present a sufficiency condition for convexity of the closed-loop QFT bounds.

1. QFT BOUNDS

Consider the QFT bounds at a single frequency. Let P be the closed set (i.e., template) where the open-loop plant is allowed to vary in: $P \in \mathsf{P}$. Let T denote the closed set describing the specs on the complementary sensitivity function: $T \in \mathsf{T}$. The design problem amounts to finding a stabilizing C satisfying, at this frequency,

$$\frac{PC}{1+PC} \in \mathsf{T} \quad \text{for all} \quad P \in \mathsf{P} \tag{1}$$

Let us introduce the mapping $f : z \rightarrow w$

$$w = f(z) = \frac{z}{1+z}$$

with its inverse $g : w \rightarrow z$

$$z = g(w) = \frac{w}{1-w}$$

Then the QFT bound B_{P_0} is

$$\mathsf{B}_{P_0} \equiv \bigcap_{P \in \mathsf{P}} \frac{P_0}{P} g(\mathsf{T}) \tag{2}$$

In practice, QFT bounds are referred to and displayed only by their boundary.

Generally speaking, QFT bounds are not necessarily convex sets. However, as we show below, mapping B_{P_0} into bounds on T_0

$$T_0 \equiv f(P_0 C) = \frac{P_0 C}{1+P_0 C} \tag{3}$$

can result in convex sets under certain conditions. The mapped bounds are the set

$$\mathsf{B}_{T_0} \equiv \{ f(P_0 C) : \; P_0 C \in \mathsf{B}_{P_0} \} \tag{4}$$

Then, $P_0 C \in \mathsf{B}_{P_0} \iff T_0 \in \mathsf{B}_{T_0}$ and

$$\mathsf{B}_{T_0} \equiv \bigcap_{P \in \mathsf{P}} f\left(\frac{P_0}{P} g(\mathsf{T})\right) \tag{5}$$

2. AUTOMATIC LOOP-SHAPING

In this section we formulate automatic loop-shaping of QFT controllers as a linear programming problem.

Problem Setup. The automatic loop-shaping problem is to find $P_0 C$

minimize $h(P_0 C)$

subject to $P_0 C(j\omega) \in \mathsf{B}_{P_0}(j\omega), \; \forall \omega$

This problem is equivalent to find T_0

minimize $h(g(T_0))$

subject to $T_0(j\omega) \in \mathsf{B}_{T_0}(j\omega), \; \forall \omega$

To perform linear programming, $\mathsf{B}_{T_0}(j\omega)$ is required to be convex. In the following section, we discuss the convexity of $\mathsf{B}_{T_0}(j\omega)$.

3. CONVEXITY OF CLOSED-LOOP QFT BOUNDS

The performance specification T is typically a closed disk of finite radius centered at the origin. Our results are based on this assumption.

The following Theorem gives a sufficient condition for convexity of B_{T_0}.

Theorem 1. *If there exists a finite complex number c such that*

$$-1 \notin \frac{c}{P} g(\mathsf{T}), \quad \forall P \in \mathsf{P} \tag{6}$$

then there exists a nominal plant P_0 *such that* B_{T_0} *is convex.*

Proof: Because the set T is a closed disk centered at the origin and g is bilinear, then $\frac{c}{P} g(\mathsf{T})$ is either a disk or the complement of a disk. Moreover, since $-1 \notin \frac{c}{P} g(\mathcal{T})$, for all $P \in \mathcal{P}$, then $\frac{c}{P} g(\mathcal{T})$ does not contain the critical point of the bilinear map f. Hence, $f\left(\frac{c}{P} g(\mathcal{T})\right)$ is a closed disk for any $P \in \mathcal{P}$.

Now taking $P_0 = c$, we have

$$\mathcal{B}_{T_0} \equiv \bigcap_{P \in \mathcal{P}} f\left(\frac{c}{P} g(\mathcal{T})\right)$$

Finally, since the intersection set of convex sets is itself a convex set, we have shown that $\mathcal{B}_{T_0}$ is convex.

Verification of (6) may not be obvious. Fortunately, we can derive an alternative result, which is based on Nichols charts and templates. Verification of (6) is then a matter of graphical observation. First let us present some notations. Let $n(\bullet)$ denote the mapping from complex plane to Nichols charts. For any set $\mathcal{Q}$ in the Nichols charts, let $\mathcal{Q}_{-1}$ denote the symmetric set of $\mathcal{Q}$ corresponding to (-180°, 0dB), and let $\overline{\mathcal{Q}}$ denote the complementary set of $\mathcal{Q}$. Then we have the following results.

Theorem 2. *If there exists a finite complex number c such that*

$$n(c)-n(\mathcal{P}) \subseteq \overline{n(g(\mathcal{T}))}_{-1} \qquad (7)$$

then there exists a nominal plant P_0 *such that* $\mathcal{B}_{T_0}$ *is convex.*

Proof. See [1].

Corollary 1. *If* $\overline{n(g(\mathcal{T}))}$ *is symmetric about the Nichols chart line* $\{(\phi,\rho) : \phi = -180^o, -\infty < \rho < \infty\}$, *condition (7) in Theorem 2 is equivalent to:*

$$n(\mathcal{P}) - n(c) \subseteq \overline{n(g(\mathcal{T}))} \qquad (8)$$

Let us illustrate our results using a simple example adapted from the QFT Control Design Toolbox in MATLAB [3]. The $n(\mathcal{P})$ at frequencies 0.1 and 100 rad/sec are shown in Fig.1. The specification is the set {T: $|T(j\omega)|<1.05$}. The QFT bounds $\mathcal{B}_{P_0}$ will not be convex. Note that the area inside the M-circle is exactly $\overline{n(g(\mathcal{T}))}$ and we can observe in Fig.1 that condition (8) is satisfied at 100 rad/sec (i.e., $n(\mathcal{P})$ could be shifted so to be contained inside the M-circle). Hence, from Corollary.1, $\mathcal{B}_{T_0}$ will be convex at 100 rad/sec if we select suitable nominal plant.

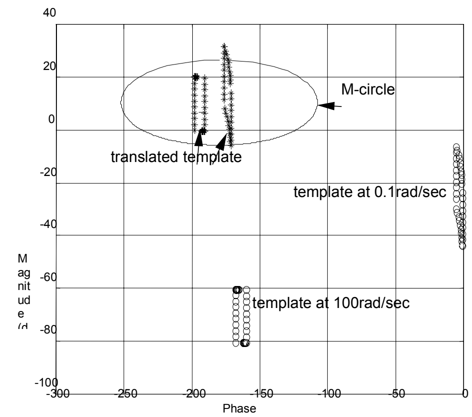

Fig 1: Templates at 0.1 and 100 rad/sec and $\overline{n(g(\mathcal{T}))}$

When considering the QFT bound for a sensitivity reduction specification

$$|S(j\omega)|<t<1 \qquad (9)$$

We have following result about the convexity of the corresponding $\mathcal{B}_{T_0}$.

Corollary 2. *For the sensitivity reduction specification (9), there always exists a nominal plant* P_0 *such that* $\mathcal{B}_{T_0}$ *is convex.*

Proof. See [1].

4.CONCLUSION

The QFT bound convexity problem in automatic loop-shaping has been investigated. New results show that by converting open-loop bounds into appropriate closed loop QFT bounds, it is possible to obtain convex bounds required for convex optimization. A sufficiency condition for the convexity of $\mathcal{B}_{T_0}$ was given.

REFERENCE

[1] Chait, Y., Chen, Q., and Hollot, C.V., "Automatic Loop-Shaping of QFT Controllers Via Linear Programming," submitted to ASME J. Dynamic Systems, Measurement, and Control.

[2] Bryant, G.F., and Halikias, G.D., “Optimal Loop-Shaping for Systems with Large Parameter Uncertainty via Linear Programming,” *Int. J. Control*, Vol. 62, pp. 557-568, 1995.

[3] Borghesani, C., Chait, Y., and Yaniv O., *The QFT Control Design MATLAB Toolbox*, The MathWorks, Inc., Natick, MA, 1995